Chapter 1

Space and Time Relationships

Metric predictions of interest in this document derive almost exclusively from data provided in the form of time-tagged positions, velocities, and related attributes of space objects over prescribed time spans. They emanate from time and space coordinate tuples that may be expressed in varied forms and viewed from diverse viewpoints. Knowledge of the interdependence of time and spatial coordinates is thus essential to the formulation of output data types.

The primary space-time relationships used in metric predictions are embodied in ephemerides of spacecraft and bodies within the solar system, as well as the ephemerides of deep space stations that serve as originators and observers of spacecraft transmissions. Then, too, these radio signals traversing deep space are susceptible to the influences of space both in propagation delay and apparent angular position. All such measurable effects are important for metric predictions and must be properly accounted for in output data types.

Until 1905, when Einstein published his first paper on (special) relativity, the domains of space and time were thought to be entirely independent. However, since that time, the world is increasingly more aware of their keen interdependence. In 1916, Einstein revolutionized the world of physics with his second theory of (general) relativity. His relativity theories together prescribe how observations of events involving space and time made in one reference frame translate into corresponding observations in other reference frames. They also deal with measures that are the same, or invariant, in all coordinate systems.

4.1 Reference Frames

A key concept in measurement and metric prediction is the reference frame, which consists of a location (or origin), a clock, and a coordinate system. Locations and motions are measured relative to such frames, and are observed to obey "the laws of physics" as they pertain to the particular frame of observation. For example, local frames on Earth experience the effects of gravity. The laws of physics in this frame describe the arc of a ball clouted across a baseball field. A ball dropped immediately falls to Earth. However, an astronaut in orbit observes almost no gravity in his immediate vicinity. Objects released from the astronaut's grasp float freely about the cabin.

Space has three dimensions, which can be quantified in Cartesian, spherical, or a number of other coordinate axes. Time has but a single dimension, *onward*. Relativity theories combine all four dimensions into a single entity called *spacetime*. The geometry of spacetime is truly 4 dimensional, for there are $4 \times 4 = 16$ relationships, determined by the frame's geometry and physical particulars, that govern the course of positions over time. These are not independent, but are interrelated via certain symmetries and constrained by the coordinate system used for quantification.

As a first step to explaining gravity, Einstein got rid of gravity, in concept, by postulating the existence of *inertial*, or *Lorentz*, or *float-free* reference frames, in which every free test particle initially at rest with respect to that frame remains at rest, and every free test particle initially in motion with respect to the frame continues its motion without change in speed or direction. Newton's first law merely defines an inertial frame of reference. As we shall later discuss, special relativity (SR) is limited to float-free frames.

In actuality, inertial reference frames do not occur in nature, but may be approximated within certain regions of space and time, within some predefined accuracy, and for given purposes. For example, a comet sweeps in from remote distances, swings close to the Sun, and returns to deep space. The motion of the comet over small portions of its trajectory can be analyzed rigorously with respect to each of a series of local free-float frames using special relativity. However, the entire trajectory cannot be modeled as one inertial frame. General relativity (GR), the theory of gravitation, tells how to describe and predict orbits that traverse a string of adjacent free-float frames. Only general relativity can describe motion in unlimited regions of spacetime.

The fact that there are no real inertial frames means that test particles do not move in what appear to be straight lines forever with respect to that frame. The concept is still useful, however, for defining reference frames. In the solar system barycentric reference frame, for example, planets, spacecraft, and other bodies move in curved paths. The reference frame is still valid for measurements and predictions, but the laws of motion must be formulated to explain why the motion of a test particle is not rectilinear. Newton's explanation was that masses exert forces on each other, and this causes trajectories to behave as observed. Einstein's explanation was that masses and energy cause time and space to be distorted, and it is this distortion that produces the observed trajectories.

A fundamental axiom of relativity theories is that observations are made relative to frames, and that phenomena may be observed separately in several frames at once. Observations in one frame may be interpreted in another frame by transformation of coordinates. However, failure to translate observations correctly or to mix observations among frames can often lead to confusion, paradox, and incorrect conclusions.

For example, suppose that a spacecraft's clock was synchronized with the local Earth clock at launch, and later in orbit transmits short time-coded pulses towards a receiver on Earth at one-second intervals. The receiver can measure the instant of arrival and can decode the incoming time stamp to learn the spacecraft clock's time stamp. But the receiver cannot expect the pulses to arrive at exactly one-second intervals according to its own local clock, even if both clocks are perfect! Interval irregularities, or differences in intervals as measured by the ground clock, are attributable to one-way Doppler effects, and may be used to infer trajectory information, if desired.

In this case, the transmitted interval is regulated by the spacecraft clock and the received interval is measured by the receiver clock. No intermixing of frame observables takes place, and the irregularities are attributed to the motions between the two frames.

But even though the spacecraft and ground receivers were synchronized at launch, the receiver cannot reliably calculate the transit light time by differencing the local time of a reception using the spacecraft's time code of transmission. In this case, the calculation differences an observable in one frame with an observable in another frame. As we shall later see, a clock in orbit, relieved of much of Earth's gravity, may run faster than the one found in the receiver.

Consequently, the clock difference computation might well yield a negative result! The relativistic method for light transit time determination is to determine the transmission and reception times using a common observation frame (e.g., an ephemeris reference frame), and then translating the difference into the observation frame of the receiver.

However, the definition of *light time* of interest in the DSN is not necessarily an observable. Rather, it is the difference between the temporal states of the transmitter and receiver, wherever they may be situated. In a one-way geometry between a spacecraft transmitter and ground receiver, this difference allows the ground station to reconstruct a profile of the spacecraft's temporal signature, which is useful in a number of applications, as discussed later in the chapter on light time and frequency prediction.

Principal frames of reference used in metric prediction are inertial ephemeris frames, local inertial frames, and local body-fixed rotating frames. As in the case of light time computation above, most calculations involving spacecraft, planets, and deep space stations on Earth are made in an ephemeris reference frame and then translated into observables in another frame. Translation among reference frames is therefore an essential part of the metric prediction process. Indeed, the characteristics of frame translations are fundamental to relativity theory. The method, called tensor analysis, is addressed later.

4.2 Measuring Distances

Quantities of interest in each frame are epochs and coordinates as they relate to durations and distances, or intervals of space and time. Relativity theories define interval measures that combine space and time intervals in such a way that a particular measurement is independent of the state of motion of the observer. The invariance of this measure among reference frames has forced the world to recognize that considerations of time cannot be separated from those of space.

In regular Newtonian physics, distances are measured as the lengths of displacements in Euclidean space. In particular, if $d\mathbf{x} = (dx^1, dx^2, dx^3)$ represents the Cartesian vector of a spatial displacement whose components bear the superscripts 1, 2, and 3, then the squared distance ds^2 is found by evaluating the familiar Pythagorean form

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1-1}$$

If, however, the coordinate system is other than Cartesian, and the distance is measured as $d\mathbf{u} = (du^1, du^2, du^3)$, in which there is a transformation of the form $x^j(u^1, u^2, u^3)$ back to the Cartesian set, then differential coordinates satisfy, by ordinary differentiation, the quadratic form

$$dx^k = \sum_{i=1}^{3} \frac{\partial x^k}{\partial u^i} du^i \tag{1-2}$$

Consequently, the distance, now expressed in u-coordinates, becomes

$$\begin{aligned} ds^2 &= \sum_{k=1}^{3} \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial x^j} du^i du^j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial x^j} \right) du^i du^j \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, du^i du^j \end{aligned} \tag{1-3}$$

where the coefficients of the differential coordinates are given by

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial x^j} \tag{1-4}$$

The matrix **g** containing these g_{ij} coefficients is called the *metric*[1] of that system. The characteristic of the distance equation above is that **g** depends on the coordinate system in that space, and, regardless of which is chosen, always produces positive displacements for physical events, and, in particular, always returns the same value for the same physical displacement.

The metric **g** of a space not only tells how to compute distances between points in that space, but it also defines the method for computation of inner (dot) products, as

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[1] Technically, this is the so-called Riemann metric, which implies a positive definite matrix. Also called the metric tensor.

$$d\mathbf{u} \bullet d\mathbf{v} \triangleq d\mathbf{u}^T \mathbf{g}\ d\mathbf{v} = \sum_i \sum_j g_{ij}\, du^i\, dv^j = g_{ij}\, du^i\, dv^j \tag{1-5}$$

The final form on the right above adheres to *Einstein's summation convention*, which is that whenever there are repeated indices in an expression, that expression is summed over all values of each repeated index. Note that indices in summations appear both as subscripts and superscripts.

Some of the properties of metrics on vector spaces are that inner products are commutative, distributive, homogeneous, and, for so-called normed spaces, positive definite:

$$\begin{gathered} \mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u} \\ (\mathbf{u} + \mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w} \\ \mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w} \\ (k\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (k\mathbf{v}) = k(\mathbf{u} \bullet \mathbf{v}) \\ \mathbf{v} \bullet \mathbf{v} \geq 0 \end{gathered} \tag{1-6}$$

For normed vector spaces, the last of these conditions is met in equality if and only if $\mathbf{v} = \mathbf{0}$.

The extension of the metric equation to include the time dimension defines displacement vectors having 4 dimensions,

$$d\mathbf{x} = (dx^0, dx^1, dx^2, dx^3) = (cdt, dx^1, dx^2, dx^3) \tag{1-7}$$

The convention of time displacement being the first dimension and its designation as dx^0 are not universal in the literature, but common. Scaling time by the speed of light c also brings uniformity in units to the spacetime vector. The appearance of time as a dimension transforms the perception of purely spatial displacements in 3 dimensions into the concept of *event* intervals in spacetime. The metric[2] $\mathbf{g}$ for such spaces, as will be discussed later, may allow spacetime distances of zero for nonzero displacement vectors. However, displacements that are traversable within light speed limitations are nonnegative.

In summary, distances are reckoned with respect to the metric of the coordinate space,

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[2] The metric in this case is called a pseudo-Riemann metric tensor, which need not be positive definite, only non-degenerate.

$$ds^2 = g_{ij}\, dx^i dx^j \tag{1-8}$$

4.3 Equations of motion

Metric predictions involve determining the states of objects in motion for which space and time coordinates have been made available in the form of ephemerides provided by JPL Navigation and space flight projects. All MPG solar system and spacecraft modeling is handled via software functions provided by the Navigation Ancillary Information Facility (NAIF). A detailed mathematical treatment of how ephemerides and state vectors are generated is therefore not appropriate for inclusion here. But a discussion of the basic process will perhaps give the reader a measure of insight into the scope and complexity of the task, and an appreciation of how spacetime considerations enter into the process. Such considerations are often important in the design of prediction data types, where certain corrections must be made based on physical considerations and output requirements. The treatment here will therefore only briefly describe the method by which the equations of motion are formulated and the means by which time profiles of target states are generated.

Let Γ denote a curve defining the motion of an object over time in some reference frame of choice. The points along the curve are defined by a vector function, such as $\mathbf{r}(t) = (x(t), y(t), z(t))$ in Cartesian coordinates. Let s denote the distance along the curve from a given starting point and time,

$$s = \int_\Gamma ds = \int_\Gamma \frac{ds}{dt} dt \tag{1-9}$$

The solution for the trajectory of the object over the "shortest" path for the given problem, called the *geodesic* path, is formulated using a calculus of variations approach (Irving and Mullineux, 1959, 366). The geodesic path must satisfy the so-called Euler-Lagrange equations, given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \qquad i \to 1,2,3 \tag{1-10}$$

for each coordinate, where here $\dot{x}^i = dx^i / dt$. Specifically, x^i and $\dot{x}^i$ are treated as independent variables insofar as partial derivatives are concerned, even though the latter is the time derivative of the former. In the present context, the function to be minimized, called the *Lagrangian*, is

$$L = ds/dt = \sqrt{ds^2/dt^2} = \sqrt{g_{ij}\dot{x}^i\dot{x}^j} \tag{1-11}$$

In the special case that L is not explicitly a function of an x^i, then the partial derivative with respect to this variable is zero. This means that the corresponding Euler-Lagrange equation can be immediately integrated with respect to t, to give the somewhat simpler equation,

$$\frac{\partial L}{\partial \dot{x}^i} = const \tag{1-12}$$

which, in the present case, yields

$$\frac{1}{L}\left(g_{ij}\dot{x}^j + \frac{1}{2}\frac{\partial g_{jk}}{\partial \dot{x}^i}\dot{x}^j\dot{x}^k \right) = const \tag{1-13}$$

In ordinary Euclidean space, spatial coordinates and time are independent variables, with $g_{ii} = 1$ and $g_{ij} = 0\ (i \neq j)$. Since time does not appear among the metric coordinates, the $\dot{x}^i$ must be interpreted as components of an instantaneous arbitrary velocity along the traverse between the endpoints of the path. The velocity components in this case satisfy the equation

$$\dot{x}^i = a^i L \tag{1-14}$$

where the a^i are constants. The velocity components all bear a constant relationship with one another, and therefore the velocity direction is always the same. That is, as is well known for Euclidean space, geodesics are straight lines.

However, in gravitational space, this is no longer true. The solutions to the Euler-Lagrange equations are much more complex in this case, and require approximation and machine computation in order to solve. An ephemeris is a tabulation of the positions of a solar system object with respect to a specified frame of reference over a specified time span. Ephemerides of the bodies and spacecraft of interest form the basis of all MPG data types. Transformations are required to convert from ephemeris frames to observation frames and to compensate and correct for various astrometric and atmospheric aberrations.

4.4 Special Relativity

The basic premises of *special relativity* (SR) are that (1) the speed of light in a vacuum is the same in all non-accelerating, non-gravitational frames, and (2) the laws of physics are the same in all such frames. In special relativity, one cannot talk about *absolute* positions or velocities, but only *relative* positions or velocities. For example, one cannot sensibly ask if a particle is at rest, only whether it is at rest relative to a given frame of reference. Transformation to another reference frame can alter the way a vector points relative to the frame, but cannot change whether two vectors point in the same direction. A key concept in special relativity is that of an inertial coordinate system in which gravitation and acceleration of the frame are absent.

In special relativity the invariant spacetime interval (formal name *Lorentz interval*, but it is common just to use the term *interval*), expressed here in Cartesian coordinates, takes the form

$$ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 \tag{1-15}$$

where c denotes the speed of light in a vacuum, dt is the interval of time, and dx, dy, and dz denote the spatial components of a displacement in the Cartesian system. The time interval is that which has elapsed during the traverse of the spatial interval. The reader will note that the magnitude dr of the displacement vector $d\mathbf{r} = (dx, dy, dz)$ is invariant under rotations and translations of the coordinate axes, and so the metric above is independent of the origin and orientation of the reference frame. The metric[3] in this case is

$$\begin{aligned} &g_{00} = 1 \\ &g_{11} = g_{22} = g_{33} = -1 \\ &g_{pq} = 0 \quad (p, q = 0,1,2,3, \quad p \neq q) \end{aligned} \tag{1-16}$$

When two events both occur at the position of a given clock, that special clock measures directly the interval between these two events. This interval is called the *proper time*, and, since there is no spatial displacement between the events, it is given by $dt = ds/c$. Here, t has been used for t to emphasize that it is proper time. Times within a frame related to displacements are called *coordinate* times to differentiate them from proper times.

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[3] This is a pseudo-Riemann metric, in that null geodesics, or those having $ds = 0$, are possible.

If observers in two separate frames of reference watch lght from an electric spark propagate outward over space, then by the fundamental axiom of special relativity, the distance that each sees the wavefront travel in an interval of time dt will be $dr = cdt$. The interval for this situation is $ds = 0$. Light thus travels along paths having zero interval, or *null geodesics*.

Proper time does not advance along null geodesics. Photons, which move along null geodesics, therefore do not observe the passage of proper time. According to their own clock, if they had one, they do not age.

Dividing proper time by coordinate time of an interval term produces the relationship

$$\frac{dt}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \tag{1-17}$$

where v is the event speed,

$$v = \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}} \tag{1-18}$$

In order that time intervals remain real and positive, it is necessary that this speed not exceed that of light in a vacuum.

4.4.1 Solving the Geodesic Equation

The Lagrangian governing the motion of bodies along geodesics is

$$L^2 = \frac{ds^2}{dt^2} = c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \tag{1-19}$$

The Lagrangian is again explicitly independent of all coordinates, and no cross-terms appear, as was also the case in the earlier Euclidean example. But, in contrast to that example, the velocity terms here are real, as time is now one of the coordinate components. Solving the Euler-Lagrange equation produces

$$\dot{x}^i = \boldsymbol{a}^i L \qquad i = 1,2,3 \tag{1-20}$$

Therefore, SR spacetime is Euclidean in the sense that geodesics are again straight lines. However, coordinate velocities are now constant, but not entirely unrestricted, since propagation must not be greater than light speed.

Further, since metric coefficients are constants, the displacement components need not be infinitesimal. Increments may be of any length, and length measure is

$$\Delta s^2 = c^2\,\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{1-21}$$

4.4.2 Lorentz Transformations

So far, the treatment of special relativity has dealt with scalar quantities, such as spatial and time increments and speeds. Even though there is state information in the 4-vector $(\Delta t, \Delta x, \Delta y, \Delta z)$, the interval equation above is the same regardless of the orientation of the axes. Information about the direction of the undertaking does not translate from one frame of reference to another. There is another formulation of the invariance that retains such information, now to be discussed.

When two reference frames are in motion relative to one another, SR relationships among time and position intervals are described by the *Lorentz transformations.* The usual formulas appearing in the literature are in rectangular coordinates with frame *F'* moving with constant speed v (a scalar) relative to frame *F* along the positive *x*-axis of *F*. The coordinate transformations are

$$\begin{aligned} \Delta x' &= g\,(\Delta x - v\,\Delta t) \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \\ \Delta t' &= g\,(\Delta t - \tfrac{v\Delta x}{c^2}) \\ g &= 1/\sqrt{1-\tfrac{v^2}{c^2}} \end{aligned} \tag{1-22}$$

However, one may generalize the above to the situation in which the frames separate at a given velocity $\mathbf{v}$ (a vector with magnitude v) in an arbitrary direction. The vector form of the Lorentz transformation is then

$$\begin{aligned} \Delta\mathbf{r}' &= \Delta\mathbf{r} + (\tfrac{g-1}{v^2}\Delta\mathbf{r}\bullet\mathbf{v} - g\,\Delta t)\mathbf{v} \\ \Delta t' &= g\,(\Delta t - \tfrac{\Delta\mathbf{r}\bullet\mathbf{v}}{c^2}) \end{aligned} \tag{1-23}$$

The first of these, the spatial transformation, quantifies the change in apparent place of an event due to its velocity relative to an observer. The second quantifies the apparent contraction or dilation of time scales in the two frames.

The relationship between event velocities in the two reference frames can now be found. First, in the unprimed frame, let $\mathbf{u} = \Delta\mathbf{r} / \Delta t$ represent the velocity vector during the interval traversal. Then the velocity vector observed in the primed frame observes the *law of combination of velocities*,

$$\mathbf{u}' = \frac{\Delta\mathbf{r}'}{\Delta t'} = \frac{\mathbf{u} + (\frac{g-1}{v^2}\mathbf{u} \bullet \mathbf{v} - g)\,\mathbf{v}}{g\,(1 - \frac{\mathbf{u}\bullet\mathbf{v}}{c^2})} \tag{1-24}$$

If $\mathbf{u}$ and $\mathbf{v}$ are collinear, the law takes on its more usual form, in which the magnitudes satisfy

$$u' = \frac{u - v}{(1 - \frac{uv}{c^2})} \tag{1-25}$$

4.4.3 Stellar Aberration

For photons emanating from objects in space and viewed on Earth, the effect on apparent place is called *stellar aberration*. If the pointing vector to the spacecraft as seen in the ephemeris frame is $\mathbf{p} = -\Delta\mathbf{r} / \Delta r$ (rays are presumed directed toward Earth, with $\Delta r = c\ \Delta t$), then the pointing vector $\mathbf{p}'$ in the Earth frame follows directly from the Lorentz transform as

$$\mathbf{p}' = \frac{g^{-1}\mathbf{p} + (\frac{1}{1+g^{-1}}\mathbf{p}\bullet\frac{\mathbf{v}}{c} + 1)\frac{\mathbf{v}}{c}}{1 + \mathbf{p}\bullet\dfrac{\mathbf{v}}{c}}$$

The stellar aberration angle Δq can be determined from the cross product, $\mathbf{p}' \times \mathbf{p} = \sin\Delta q$. A Taylor series expansion of this produces

$$\sin(\Delta q) = \left(\frac{v}{c}\right)\sin(q) - \frac{1}{4}\left(\frac{v}{c}\right)^2 \sin(2q) + O\left(\left(\frac{v}{c}\right)^3\right) \tag{1-26}$$

where q is the angle between $\mathbf{p}$ and $\mathbf{v}$. A Taylor series expansion of the apparent pointing vector produces the approximate result

$$\mathbf{p}' = \mathbf{p}\left(1 - \frac{\mathbf{p}\bullet\mathbf{v}}{c} - \frac{v^2}{2c^2} + \left(\frac{\mathbf{p}\bullet\mathbf{v}}{c}\right)^2\right) + \frac{\mathbf{v}}{c}\left(1 - \frac{\mathbf{p}\bullet\mathbf{v}}{2c}\right) + O\left(\left(\frac{v}{c}\right)^3\right)$$
$$\mathbf{p}' \approx \mathbf{p}\left(1 - \frac{\mathbf{p}\bullet\mathbf{v}}{c}\right) + \frac{\mathbf{v}}{c} \qquad (1\text{-}27)$$

The aberration angle of the simpler approximation is merely

$$\sin(\Delta q) = \left(\frac{v}{c}\right)\sin(q) \qquad (1\text{-}28)$$

which is in error by

$$e(\Delta q)_{approx} = \sin(\Delta q)_{approx} - \sin(\Delta q)_{SR} = \frac{1}{4}\left(\frac{v}{c}\right)^2 \sin(2q) \qquad (1\text{-}29)$$

For observers on Earth, the simpler approximation error is less than 0.15 µdeg error, and there is little reason to normalize the result, since the departure of the magnitude from unity is at most about 1×10^{-8}. However, for utmost accuracy, the result should be normalized.

The classical Newtonian expression for the direction of a source as seen by a moving observer is

$$\mathbf{p}'_{Newton} = \frac{\mathbf{p} + \frac{\mathbf{v}}{c}}{\left|\mathbf{p} + \frac{\mathbf{v}}{c}\right|} \qquad (1\text{-}30)$$

The Newton aberration angle, computed as above, is

$$\sin(\Delta q) = \left(\frac{v}{c}\right)\sin(q) - \frac{1}{2}\left(\frac{v}{c}\right)^2 \sin(2q) + O\left(\left(\frac{v}{c}\right)^3\right) \qquad (1\text{-}31)$$

The error in the Newton formula is therefore

$$e(\Delta q)_{Newton} = \sin(\Delta q)_{Newton} - \sin(\Delta q)_{SR} = -\frac{1}{4}\left(\frac{v}{c}\right)^2 \sin(2q) \qquad (1\text{-}32)$$

The classical Newtonian aberration is thus also within 0.15 µdeg of the SR result for Earth observers. There is therefore little to recommend one formula over the other insofar as accuracy is concerned, as all are well within MPG requirements. The Newtonian form is perhaps the easiest to calculate and has unit magnitude within machine precision. The SPICE routine `STELAB` is available for this purpose.

4.4.4 Light Transit Time Considerations

Planetary aberration refers to the apparent displacement due to the motion of an observed object. It relates the coordinates of two sets of position coordinates to the coordinate time required for light to travel between the two positions. When a photon is transmitted from an Earth station, having position $\mathbf{r}_1(t_1)$ at time t_1, is subsequently received at a spacecraft, having position $\mathbf{r}_2(t_2)$ at time t_2, as observed in a common frame of reference, then the difference $t_2 - t_1$ is the transit time of light between the two positions, or *light time*. The basic premise of SR is that

$$t_2 - t_1 = \frac{|\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)|}{c} \tag{1-33}$$

Ephemerides of the station and spacecraft provide positions versus time, but only when those times are known ahead of time. If only one of the time instants is given, the expression above becomes an implicit equation to be solved for the other.

The solution of this *light time equation* requires an iterative approach. The following supposes that t_1 is fixed and the goal is to find t_2 via a series of approximations $\hat{t}_{2,i}$, in which

$$\hat{t}_{2,i+1} = t_1 + \frac{|\mathbf{r}_2(\hat{t}_{2,i}) - \mathbf{r}_1(t_1)|}{c} \tag{1-34}$$

A similar approach works when t_2 is the given parameter. The first approximation is to set the estimate $\hat{t}_{2,1} = t_1$. The convergence rate can be approximated using the triangle inequality and a first-order Taylor series approximation, to produce

$$
\begin{aligned}
|t_2 - \hat{t}_2|_{i+1} &= \frac{|\mathbf{r}_2(t_2) - \mathbf{r}_1(t_1)|}{c} - \frac{|\mathbf{r}_2(\hat{t}_2) - \mathbf{r}_1(t_1)|}{c} \\
&\le \frac{|\mathbf{r}_2(t_2) - \mathbf{r}_2(\hat{t}_2)|}{c} \approx \frac{|\dot{\mathbf{r}}_2(\hat{t}_2)|}{c} |t_2 - \hat{t}_2|_i
\end{aligned}
\tag{1-35}
$$

Solar system velocities are generally below 50 km/s and distances are less than 100 AU. Therefore, convergence requires no more than 3 iterations to reach accuracy greater than found in JPL ephemerides.

The SPICE routine `LTIME` computes uplink and downlink light times using this method.

4.5 General Relativity

The theory of *General relativity*, which Einstein published in 1915, adds to SR the axiom (the so-called *equivalence principle*) that accelerating frames cannot be distinguished from frames in which the gravitational field exhibits the same mathematical profile as the acceleration of the other frame. In GR, one can no longer speak of relative states, except when they lie at the same point in space and time. The reason for this is that, in order to compare them, one state would have to be moved over to the other along some path (parallel transfer) without turning it or stretching it, which turns out to be infeasible due to the curvature induced by gravitating objects and/or accelerating frames. Thus, the concept of an inertial frame no longer makes total sense. Nevertheless, approximations to inertial frames are useful in defining certain reference frames.

GR's governing relationships, called the *Einstein's field equations*, are believed by many to be the most fundamental laws describing the structure and development of the universe. To most non-experts, however, the subject lies shrouded in mystery, confusion, and daunting mathematical complexity. Mere representation of the theory relies not just on vector calculus, as does special relativity, but upon differential geometry, tensor algebra, non-Euclidean manifolds, and physical interrelation of energy and matter.

Fortunately, in the solar system, the departures of predictions involving relativistic events from their Newtonian counterparts are only slight. This enables the use of classical methods, with subsequent corrections for relativistic effects. This chapter indicates the general basis for such computations, but quickly focuses on simpler models that apply to predictions of motions of objects within the universe.

4.5.1 Tensors

Because they appear profusely in GR theory, a short treatment of *tensors* is included here. A treatise on tensors is not appropriate, but a definition of what they are and how they are used will perhaps help clear some of the haze that often surrounds the term. Both subscripts and superscripts are used in tensor notation to number coordinate components. The particular usage, subscript or superscript, depends on the transformational characteristics of the entity.

Tensors are used to describe differential invariants involving transformations from one set of coordinates to another. If $x^1, x^2, \ldots, x^n$ represents a set of generalized coordinate axes (the integers are superscripts, not powers), if $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ represents another set, if a function $T = T(x^1, x^2, \ldots, x^n)$ transforms into the function $\bar{T} = \bar{T}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ under this coordinate transformation, and if the transformation follows the rule

$$\bar{T}^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots bs} = \left|\frac{\partial x}{\partial \bar{x}}\right|^N T^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots bs} \frac{\partial \bar{x}^{a_1}}{\partial x^{a_1}} \frac{\partial \bar{x}^{a_2}}{\partial x^{a_2}} \cdots \frac{\partial \bar{x}^{a_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \bar{x}^{b_1}} \frac{\partial x^{b_2}}{\partial \bar{x}^{b_2}} \cdots \frac{\partial x^{b_s}}{\partial \bar{x}^{b_s}} \quad (1\text{-}36)$$

where the various indices run through the integers 1, 2, …, n, then T is a tensor. The terms above adhere to Einstein's summation convention wherein repeated indices in an expression are summed over all values of each repeated index. The tensor definition above therefore represents a summation of n^{r+s} terms.

The exponent N of the Jacobian $|\partial x / \partial \bar{x}|$ is called the weight of the *tensor* field. If $N = 0$, the tensor field is said to be *absolute*. If $s = 0$, the tensor is called purely *contravariant*, and if $r = 0$, it is purely *covariant*. Otherwise, it is called a *mixed* tensor. Tensors are said to be of the same kind if they have the same number of contravariant indices (superscripts) and covariant indices (subscripts) and are of the same weight. Contravariant tensors are those in which differential terms in the transform appear as $\partial x^{new} / \partial x^{old}$; covariant tensors are those in which this contribution is inversely related.

As points of information, the tangent vector along a spacetime curve at a given point in time is an absolute contravariant tensor of order 1, whereas the vector Δf

representing the gradient of a scalar function is an absolute covariant tensor of order 1. The metric g_{ij} is an absolute covariant tensor of order 2. and is commonly referred to as the *metric tensor* of the space.

Since relativity theories treat invariants in spacetime, it is natural that tensors, which relate to invariants under transformations of coordinates, are brought to bear in the analyses.

4.5.2 Einstein's Field Equations

Generally, gravitational effects are interrelated with a given distribution of mass, momentum, and energy by a system of 10 coupled hyperbolic-elliptic nonlinear partial differential equations, called the *Einstein field equations*. This system, the fundamental formula governing dynamics within a gravitational field, is given by

$$G_{mn} = 8p\, G T_{mn} \tag{1-37}$$

where T_{mn} is the stress-energy tensor (also called the energy-momentum tensor), G is Newton's constant, and G_{mn} is the so-called Einstein tensor. The subscripts m and n range from 0 to 3 and designate time and spatial dimensions. The equation applies the Einstein convention whereby quantities are summed over repeated indices. Even though there are $4 \times 4 = 16$ elements in the tensors, symmetry reduces the number to 10 different ones, and the choice of coordinate system puts constraints on 4 of these, so there are in reality only 6 independent degrees of freedom.

The left-hand side of the equation relates to spacetime curvature and the right-hand side describes spacetime mass, momentum, and energy. The nonlinearity of the Einstein field equations stems from the way that masses distort the geometry of the space in which they reside.

The complexity of the theory lies in the tensors G_{mn} and T_{mn}. The former is based in the mathematics of differential geometry, which quantifies curvature measures on a generalized space. The latter is grounded in physics, which defines how forces, masses, momentum, and position interrelate.

The left-hand side, the Einstein tensor, expresses curvature relationships in terms of the metric tensor. Given a metric tensor and coordinate system, the Einstein tensor can then be computed (not easily, but straightforwardly). Unfortunately, this is not the usual situation, which is that the stress-energy tensor is given, and the metric tensor must be found. Theoretically, one is able to determine the metric tensor that characterizes the space curvature from these equations.

The fundamental insight of the theory, as embodied in this equation, is that mass distorts the geometry of spacetime and characterizes this distortion in the form of a metric tensor. The metric tensor, which now embodies the geometry of spacetime, in turn, governs how masses move via the Euler-Lagrange equations, which yield the equations of motion.

But the complexity of even writing out the expressions above in symbolic form is daunting. In fact, these equations appeared to be so complicated that when Einstein first formulated them in 1915, he did not believe that a solution would ever be found. He was therefore quite surprised when, only a year later, Karl Schwarzschild discovered a solution for the case of a static, spherically symmetric metric. For a point mass this solution is known as the Schwarzschild solution. Schwarzschild later also found an exterior solution for the case of a gravitating spherical body with constant density.

4.5.3 General Relativity Metric Tensor

Based on Schwarzschild's work, a solution to the field equations for the case of a massless particle moving in the gravitational field of n massive bodies was obtained by J. Droste in 1916. Only a little later, W. deSitter extended Droste's work to account for the mass of the body whose motion is desired, but erred slightly in one term. The method was corrected by Eddington and Clark in 1938. The resulting components of the metric tensor for the n-body problem are (see Moyer, 1971, 6)

$$
\begin{aligned}
g_{11} &= g_{22} = g_{33} = -\left(1 + \frac{2}{c^2}\sum_{j \neq i}\frac{m_j}{r_{ij}}\right) \\
g_{pq} &= 0 \quad (p, q = 1,2,3, \quad p \neq q) \\
g_{0p} &= g_{p0} = \frac{4}{c^3}\sum_{j \neq i}\frac{m_j \dot{x}_j^p}{r_{ij}} \quad (p = 1,2,3) \\
g_{00} &= 1 - \frac{2}{c^2}\sum_{j \neq i}\frac{m_j}{r_{ij}} + O\left(\frac{1}{c^4}\right)
\end{aligned}
\qquad (1\text{-}38)
$$

where the index i refers to the body whose motion is desired, j refers to each of the remaining bodies, $m_j = Gm_j$ is the gravitational constant of body j whose rest mass is m_j, and r_{ij} is the coordinate distance between bodies i and j. The coordinate system is presumed to be a non-rotating inertial frame whose origin is at the barycenter of the system.

It is perhaps interesting to note at this point that the summation appearing in each of the g_{ii} terms is the sum of the Newtonian gravitational potentials of the ensemble of masses at the body whose motion is being addressed. It is sometimes denoted in the literature by U_i. The metric equation

$$ds^2 = g_{ij}\,dx^i dx^j \qquad (1\text{-}39)$$

governs how distances are measured in relativistic space.

4.5.4 Ephemeris Generation

Solution of the Euler-Lagrange equations using the GR metric tensor values given above lead to lengthy expressions for the acceleration of each body in the system. Details of the derivation may be found in [Moyer1971], Appendix C. The short form of the solution is

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}\left[1 + O\left(\frac{1}{c^2}\right)\right] + O\left(\frac{1}{c^2}\right) \qquad (1\text{-}40)$$

The first term of this relationship is the Newtonian acceleration of body i. The other terms are GR perturbations deriving from the metric tensor elements. The entries that appear among the $O(1/c^2)$ terms are recognizable as members of the metric tensor $\mathbf{g}$, but are suppressed here for brevity. JPL's Double Precision Orbit Determination Program (Moyer 1971) performs simultaneous numerical integration of these vector equations over the n objects of the system to produce ephemerides for all the bodies included. The reader desiring complete details may consult the references. Further discussion appears in the chapter on Celestial Mechanics.

4.5.5 Lorentz Transformations

There is a GR equivalent of the Lorentz transformations between isotropic Cartesian coordinate systems, in which the metric tensor terms satisfy

$$\begin{aligned} g_{00} &= l^2 \\ g_{ii} &= -k^2 \quad i = 1,2,3 \\ g_{ij} &= 0 \qquad i \neq j \end{aligned} \tag{1-41}$$

One may verify by substitution that the transformation between two such frames, given by

$$\begin{aligned} k'\,d\mathbf{r}' &= k\,d\mathbf{r} + \left(k\,\frac{g-1}{v^2}(d\mathbf{r}\bullet\mathbf{v}) - g\;l\;dt \right)\mathbf{v} \\ l'\,dt' &= g\left(l\;dt - k\,\frac{d\mathbf{r}\bullet\mathbf{v}}{c^2} \right) \end{aligned} \tag{1-42}$$

satisfies the interval condition

$$ds^2 = g_{00}\,c^2 dt^2 + g_{11}\,dr^2 = g'_{00}\,c^2 dt'^2 + g'_{11}\,dr'^2 \tag{1-43}$$

Note now, however, that the GR Lorentz transformations hold only for infinitesimal intervals, whereas, the SR transformations held for intervals of any length. Further, the coordinate speed of light is no longer c, as the condition for a null geodesic is that

$$v_c = \frac{dr}{dt} = c\sqrt{-\frac{g_{00}}{g_{11}}} = c\,\frac{l}{k} \tag{1-44}$$

For the n-body metric tensor given earlier,

$$\begin{aligned} k &= \sqrt{1 + \frac{2U}{c^2}} \\ l &= \sqrt{1 - \frac{2U}{c^2}} \\ U &= \sum_{j \neq i} \frac{m_j}{r_{ij}} \end{aligned} \tag{1-45}$$

A Taylor series expansion of the coordinate speed of light, in which terms of order smaller than U^2 / c^4 are ignored, gives

$$v_c = c\left(1 - \frac{2U}{c^2} + \cdots\right) \tag{1-46}$$

The coordinate speed of light thus decreases slightly as photons traverse regions of high gravitational potential. The Newtonian light time between two points is the geometric coordinate distance divided by c. However, the actual light time will be slightly longer because the speed of light is decreased somewhat, and because the light path is no longer straight, but curved, due to gravitational deflection.

4.5.6 Stellar Aberration

The GR stellar aberration formula follows from the generalized Lorentz transformations by setting $\mathbf{p} = -d\mathbf{r}/dr$ (rays are presumed directed toward Earth, with $dr = c\,dt\,l/k$). The resulting pointing vector $\mathbf{p}'$ in the Earth frame turns out to be exactly the same form as that given in the SR case. Therefore, corrections in a GR frame due to observer velocity are the same as used in an SR frame.

4.5.7 Gravitational Deflection

Light propagation under GR is a combination of two effects, amounting to stellar aberration and curvature effects of gravity. But SR and GR stellar aberration formulas the same. Angular corrections due to gravitational distortions can then be added in as a separate and independent operation. Although GR light time effects derive from the same geodesic analysis as do angle compensations, they are treated separately, in the next section.

Gravitational deflection of electromagnetic rays was predicted by Einstein and confirmed at the eclipse of in 1919, and many times since with ever-increasing accuracy. Murray (1981) developed a formula for it based on earlier solutions of the geodesic equation by Eddington, which, in turn, stemmed from the Schwarzschild solution. The deflection increases as the ray path comes closer to the Sun, and drops off rapidly as geocentric elongation increases.

For observers on Earth, the angle Δf between the geometric observer-target pointing vector $\mathbf{p}$ and the tangent vector $\mathbf{p}'$ to the observer-target light path is given by following formula appearing in the *Explanatory Supplement to the Astronomical Almanac*,

$$\Delta f = \frac{2m}{c^2 E}\tan(y\ /2) = \frac{2m}{c^2 E}\,\frac{\sin(y\ )}{1+\cos(y\ )} \qquad (1\text{-}47)$$

where E is the distance of the observer from the deflecting mass (here the Sun), $m = Gm_S$ is the Sun's gravitational constant, and y is the heliocentric elongation (angle at the Sun between Earth and target). Since light is deflected toward the Sun, the ray reaching the Earth observer appears to emanate from an angle Δf from $\mathbf{p}$ away from the Sun. The limiting elongation occurs when the target disappears behind the Sun, which is about 180°-0.25°; in this case the maximum deflection is about 1.866 arcsec.

If $\mathbf{e}$ and $\mathbf{q}$ respectively denote the heliocentric Earth and target pointing vectors, then the vector cross product $\mathbf{q}\times\mathbf{e}$ is perpendicular to the Sun-Earth-target plane and has magnitude $\sin(y\ )$. It then follows that the vector $(\mathbf{q}\times\mathbf{e})\times\mathbf{p}/\sin(y\ )$ is a unit vector in the Sun-Earth-target plane perpendicular to $\mathbf{p}$ and away from the Sun. The laws of vector algebra allow the triple vector product (see Lass 1950) to be expressed in the readily calculable form

$$(\mathbf{q}\times\mathbf{e})\times\mathbf{p} = (\mathbf{p}\bullet\mathbf{q})\,\mathbf{e} - (\mathbf{e}\bullet\mathbf{p})\,\mathbf{q} \qquad (1\text{-}48)$$

The apparent light path direction, by simple vector addition, is therefore

$$\mathbf{p}' = \mathbf{p} + \frac{\tan(\Delta f)}{\sin(y\ )}(\mathbf{q}\times\mathbf{e})\times\mathbf{p} = \mathbf{p} + \frac{g_1}{g_2}\left[(\mathbf{p}\bullet\mathbf{q})\mathbf{e} - (\mathbf{e}\bullet\mathbf{p})\mathbf{q}\right] \qquad (1\text{-}49)$$

in which the coefficients g_1 and g_2 are

$$\begin{aligned} g_1 &= \frac{\tan(\Delta f)(1+\cos(y\ ))}{\sin(y\ )} \approx \frac{2m}{c^2 E} \\ g_2 &= 1 + \mathbf{q}\bullet\mathbf{e} = (1+\cos(y\ )) \end{aligned} \qquad (1\text{-}50)$$

The approximation for g_1 holds because the maximum deflection is very small. The value of g_1 is always close to 1.9743×10^{-8}, since Earth-Sun distance is fairly constant, but g_2 varies between 0 and 2. The vector $\mathbf{p}'$ is a unit vector within a magnitude error of $(\Delta f)^2/2 \approx 4\times 10^{-11}$.

The above formulation of deflection is the result of a first-order development that assumes small deviations in the photon track from a straight line in Euclidean space. An isotropic metric has been assumed, and only the Sun's gravitational field has been included. Each of the planets causes a similar effect that is smaller by a factor equal to the ratio of the planet's mass to that of the Sun (1/1047 for Jupiter). The errors in neglecting second-order effects and from using heliocentric, rather than barycentric, measurement of distances is cited to be less than about 0.2 mdeg.

4.5.8 Light Time Compensation

Accurate solutions to the light time equation have only become necessary with the advent of spacecraft probes of the solar system and accurate instruments, such as the Deep Space Network, that can detect even slight deviations in path lengths. A principal effect in GR is the additional propagation delay due to spacetime curvature. Masses of influence not only lengthen the path over this traverse due to curvature, but also slow the coordinate velocity of light as observed in the reference frame.

The GR solution appears in (Moyer 1971, Appendix C). Below, t_i and t_j represent the times of transmission and reception, respectively, and r_{ij}, r_{ik}, and r_{jk} denote distances between transmitter and receiver and between transmitter or receiver and body k. The light time is

$$t_j - t_i = \frac{r_{ij}}{c} + \frac{2}{c^3} \sum_k m_k \left(\ln\left(\frac{r_{ik} + r_{jk} + r_{ij}}{r_{ik} + r_{jk} - r_{ij}} \right) + \frac{4 r_{ij} m_k}{((r_{ik} + r_{jk})^2 - r_{ij}^2) c^2} \right) \quad (1\text{-}51)$$
$$+ O(1/c^7)$$

This equation is evaluated and returned by the NAIF `RLTIME` utility, which also returns the derivative of the light time with respect to reception time.

References

[Atkinson1963] Atkinson, Robert d'Escourt, "General Relativity in Euclidean Terms", *Proceedings of the Royal Society of London,* Series A, Mathematical and Physical Sciences, Volume 272, Issue 1348, pp. 60-78, 1963.

[Irving & Mullineux1959] Irving, J., and Mullineux, N., *Mathematics in Physics and Engineering*, Academic Press, New York, 1959.

[Lass1050] Lass, Harry, *Vector and Tensor Analysis*, McGraw-Hill Book Company, Inc., New York, 1950.

[Moyer1971] Moyer, T. D., "Mathematical Formulation of the Double Precision Orbit Determination Program,", Technical Report 32-1527, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, May 15, 1971.

[Moyer2000] Moyer, T. D., *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation*, JPL Publication 00-7, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, October 2000.

[Murray1981] Murray, C. A., "Relativistic Astrometry," Royal Astronomical Society, Monthly Notices, Volume 195, May 1981, pp. 639-648.

[Seidelman1992] Seidelmann, K. P. (ed), *Explanatory Supplement to the Astronomical Almanac*, University Science Books, Mill Valley, CA,1992.

[Taylor1992] Taylor, E. F., and Wheeler, J. A., *Spacetime Physics*, Introduction to Special Relativity, Second Edition, W. H. Freeman and Company, 1992.