Chapter **2**

Time Scales, Epochs, and Intervals

Time is an easy concept to envision. Simply put, it is a nonspatial continuum in which events occur in apparently irreversible succession from the past through the present to the future. However, the term is also used to describe an interval separating two points on this continuum, or duration. Or it can refer to the length of this duration. It can be a number representing a specific point on this continuum, reckoned in hours, minutes, seconds, or in some other scale of measurement. It may denote a system by which such intervals are measured or such numbers are reckoned, as solar time. It may define a season (e.g., harvest time) or an era in history (e.g., war time).

Time can therefore be a daunting concept to grasp and quantify, just because there are so many ways in which it is defined, measured, and applied. Even in scientific usage, the term spans a large repertoire of contextual applic ations.

The attributes of time that are important in metric prediction are

1. Epoch, or instant.
2. Interval, defined by two instants.
3. Duration, or interval length
4. Scale, or unit of duration measure
5. System, or application context

Because a number of timescales and reference systems apply to metric prediction generation, conversions among systems of measure are also of great importance.

2.1 Epochs

An *epoch* is an arbitrary fixed instant of time or date used as a chronological reference datum for calendars, celestial reference systems, star catalogs, and orbital motions. The standard epoch used within the MPG is noon on January 1, 2000, otherwise known as J2000, which is the start of the Julian year 2000. The Julian Date of this epoch is JD 245,1545.0. Other instants of time within the MPG are calculated as interval points having duration measured in seconds past J2000. Instants before J2000 have negative numeric values.

2.2 Calendars

A *calendar* is a system of organizing units of time (e.g., days) for the purpose of reckoning time over extended periods. By convention, the day is the smallest calendrical unit (the measurement of fractions of a day is classified as *time keeping*). Actually, "day" here means a day and a night. To avoid ambiguity whether "day" means "the daylight hours" or "a day and a night," calendricists use the word *nychthemeron* (pl. *nychthemera*) for the 24-hour period spanning a day and a night. Generally, "days" as used in this work refers to nychthemera.

According to a 1987 estimate cited in the *Explanatory supplement to the Astronomical Almanac* (ESAA), there are about forty calendars used in the world today. This chapter, however, is limited to only those appear in MPG computations, with a short history or description of each. For a fuller description, consult the ESAA ([Seidelmann1992]).

Early calendars commonly counted years from the beginning of the rule of a King, Emperor, or leader (regnal years). The Romans counted from the start of the reign of the Emperor or Caesar and reset to one when the next Emperor took over. Alternatively, they counted from the founding of Rome, and so indicated by the letters AUC (*ab urbe condita*).

The Julian calendar was introduced by Julius Caesar, in what we now denote as 45 BC, as a replacement for the more complicated Roman calendar, taking force in 45 BC (709 AUC). The calendar consisted of 3 years of 365 days followed by one having 366 days. But years continued to be counted as regnal years.

In about AD 523, the monk Dionysius Exiguus (Denis the Little) devised a way to implement rules set forth by the Nicean council (the so-called "Alexandrine

Rules") for calculating Easter. In his calculations, he chose to number the years since the birth of Christ, rather than the accession of the current monarch. He (wrongly) fixed Jesus' birth at 25 December 753 AUC, thus making the Christian era start with AD 1 on 1 January 754 AUC. How he established the year of Christ's birth is not known, although a considerable number of theories exist. He proposed this system of counting, but it was not immediately accepted.

When another monk, The Venerable Bede (AD 673-735), wrote his history of the early centuries of Anglo-Saxon England, he promoted the system of Dionysius. Its use spread during the Middle Ages until it became *a de facto* standard. Bede himself seems to have instituted the "BC" and "AD" year naming convention. In academic historical and archaeological circles, particularly in the United States, the AD period is sometimes referred to as the Common Era (CE) and the BC period as Before the Common Era (BCE). The AD-BC convention will be used in this work.

While it is increasingly common to place AD after a date, by analogy to the use of BC, formal English usage adheres to the traditional practice of placing the abbreviation before the year, as in Latin (e.g., 100 BC, but AD 100).

The Julian calendar based on the AD 1 epoch was in common use until the AD 1500s, when countries started changing to the Gregorian calendar. Some countries (for example, Greece and Russia) used it into this century, and the Russian Orthodox Church still uses it, as do some other Orthodox churches. However, the mean year in the Julian calendar was a little too long, causing the Vernal equinox to slowly drift.

The Gregorian calendar is a modification of the Julian calendar that was first proposed by the Neapolitan doctor Aloysius Lilius, and then authorized by Pope Gregory XIII, for whom it was named, on February 24, 1582. The papal bull was signed in AD 1581, but, for unknown reasons, was not printed until 1 March in 1582. It was devised to correct the equinoctial slippage and to bring calendar dates back into alignment with equinoctial phenomena and to correct the method by which leap years were calculated. The standard civil calendar in most countries today is the Gregorian calendar.

Dates that occur prior to the adoption of a calendar system may still be reckoned according to that system by *prolepsis*, or the anachronistic representation of something as existing before its proper or historical time. Thus, it is possible to determine the actual year that was 46 BC, even though no calendar contained that year.

2.3 Julian Date

Astronomers now commonly designate calendar dates by *Julian Date* (JD), which is the interval of time in days and fraction of day since the epoch 4713 BC January 1, Greenwich noon, according to the Julian proleptic calendar. The convention was devised by John F. Herschel (son of astronomer William Herschel) in 1849. He chose he beginning year as one in which the number in each of three subordinate calendar cycles[1] was unity, and because it predated all historical dates at that time.

Astronomers adopted Julian Dates in the late 19th century, but established the meridian of Greenwich as the datum instead of Herschel's use of Alexandria, since the former had been made the Prime Meridian by international conference in 1884.

Formulas for conversion between Gregorian dates and their corresponding Julian Dates (for JD > 0) appear in the ESAA (12.92). Computer programs for these conversions also appear in *Numerical Recipes* (1.1).

The *modified Julian Date* (MJD) is defined as the Julian Date minus 2400000.5. Thus J2000 is MJD 51544.5.

The term "Julian date" is also used in many applications to refer to a date format that combines the year and the number of days since the beginning of the year. Depending on the usage, the year is represented by either 2 or 4 digits, and the day of year by 3 digits. For example, January 1, 2007 is represented either as 2007001 or 07001. August 24,1999 is stored as 1999236 or 99236, since August 24 is the 236th day of the year.

This form is not based on the Julian calendar, nor is it the Julian Date discussed above. To avoid confusion and ambiguity, the term is not further used in this

[1] The cycles are indiction cycle, Metonic cycle, and solar cycle. The indiction cycle is a Roman tax cycle of 15 years declared by Constantine the Great. The Metonic cycle is a particular approximate common multiple of the tropical year and the synodic month, or 19 tropical years, which differs from 235 synodic months only by about 2 hours. The solar cycle of 28 days is related to lunation rate.

work. Should it be necessary to refer to dates in this format, the term *year-doy number* will be applied.

2.4 Time Scales

There are many time scales that are used in generating metric predictions for deep space missions applications. These may be categorized as civil time, solar time, proper time, and coordinate time. Each of these is related to the others, and there are subtle subclassifications within its category. For example, civil time is designated as Universal Time, or UT; but UT comes in a number of flavors, such as UTC, UT0, UT1, and UT1R, discussed later in this Chapter. Figure 2-1 enumerates the various timescales covered in this Supplement, along with the means of conversion among them.

2.5 International Atomic Time (TAI)

Proper time on Earth is reckoned in units of the SI (Système International) second, defined as the duration of 9,192,631,770 cycles of radiation corresponding to the transition between two hyperfine levels of the ground state of cesium 133, as measured on the geoid (mean sea level). This definition, though precise, is not totally accurate, as some variation among clocks implementing this standard is inevitable.

International Atomic Time (TAI, from the French *Temps Atomique International*), is a practical implementation of the standard that conforms as closely as technology now permits to the definition of the SI second. It is calculated as a weighed average of timescales obtained from a number of individual commercial atomic time-standards and primary frequency standards in many countries, as directed by the *Bureau International des Poids et Measures* in Sèvres, France. Corrections are applied for known effects to maintain accuracy, and the adjusted timescale is published as TAI.

As will be shown later in this chapter, velocity and gravitational potential at a given location affect the rate of a clock's oscillator. This fact causes clocks at the equator to perform differently than the same clock at a pole, due to the differences in rotational velocity and gravitational potential. For this reason, comparisons of times from various locations require that a coordinate reference frame and a set of comparison standards need to be established.

In 1980 the Consultative Committee for the Definition of the Second (CCDS) proposed to the International Committee of Weights and Measures (CIPM, in French word order) that TAI be specifically defined as a coordinate timescale at a geocentric datum line having as its unit the SI second, as obtained on the geoid in rotation, and that TAI at other locations near the geoid be extended by applying corrections for relativistic effects.

2.6 Dynamical Time (Ephemeris Time)

Dynamical time represents the independent variable of the equations of motion of celestial bodies, spacecraft, and light rays in the solar system, measured in SI seconds relative to an established epoch and inertial coordinate reference frame. It is the time coordinate of ephemerides of such entities.

Two principal inertial reference frames are used by the MPG: the solar system barycentric frame and the terrestrial geocentric frame. According to relativity theory, there must exist mathematical transformations that correspond phenomena that are observed in the two frames. The timescales of the two frames, therefore, cannot both be unique. However, they may be chosen in such a way that the timescales may differ only by periodic variations. This will be addressed sections to follow.

2.7 Terrestrial Time

The dynamical timescale for apparent geocentric ephemerides was selected by the IAU to be a unique proper timescale, from which others would be derived, and termed Terrestrial Dynamic Time, or TDT. They later decided to drop the "Dynamical" part of the name, to define the timescale as Terrestrial Time, or TT.

They decided that the TAI instant January 1.0, 1977 would be made equal to the TT instant January 1.0003725, 1977. This introduced a difference of exactly 32.184 s between the two timescales. The unit of the timescale was the SI second at mean sea level.

They also decided that a related barycentric timescale would be defined such that no periodic variations between that time scale and TT would exist.

TT is defined generally to be in step with TAI in order to take advantage of the direct availability of Coordinated Universal Time, or UTC, which is also based on the SI second measured on the geoid. But since TAI is a statistical timescale based on average observed times and TT is an idealized uniform timescale, the two are not totally consistent. The current definition is

$$TT = TAI + 32.184\ s \tag{2-1}$$

but may me altered in the future, if deemed appropriate.

The relationship between TT and TAI provides continuity with an existing "ephemeris time" that was the independent variable in ephemeris generation in that era. The chosen offset above is equal to the estimated difference between TAI and ephemeris time at the time TT was introduced.

2.8 Barycentric Coordinate Time

Proper distance in any frame conforms to the equation

$$ds^2 = g_{ij}\, dx^i dx^j \tag{2-2}$$

established in Chapter 1, where ds lies along the geodesic and dx^i are coordinate differentials along the path of travel. Substituting the g_{ij} components of the metric tensor for the n-body problem given in the previous chapter (also see Moyer, 1971, 6), using a Cartesian coordinate system, inserting the coordinate velocity, which is

$$v^2 = \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} \right) \tag{2-3}$$

and retaining only terms having order greater than $1/c^2$, the following equation results, relating proper time at some point on Earth to coordinate time in the barycentric reference frame:

$$\frac{d\mathbf{t}^2}{dt^2} = \frac{ds^2}{c^2 dt^2} = \left(1 - \frac{2U}{c^2}\right) - \left(1 + \frac{2U}{c^2}\right)\frac{v^2}{c^2} \tag{2-4}$$

Here U is the sum of the Newtonian gravitational potentials of the ensemble of masses at the body whose motion is being addressed. As is usual in relativistic time transformations, proper time runs slower than coordinate time, as evidenced by the right side of the equation being less than unity by an amount of order $(2U + v^2)/c^2$.

If proper time is being measured by a TT clock on the geoid, then the velocity above is that of the clock with respect to the barycenter, and the significant gravitational potential terms are those due to the Sun, Moon, and each of the planets (including Earth). The barycentric timescale is called *Barycentric Coordinate Time*, or TCB. Since TCB runs at a higher rate than TT, over centuries the two drift farther and farther apart.

2.9 Barycentric Dynamic Time

Earlier it was mentioned that coordinate times could be made to differ from TT by only periodic terms. Such a timescale would not then drift apart from Earth clocks in the way that TCB does. It is defined by scaling spacetime coordinates by a constant factor $\ell = 1 + L$ greater than unity chosen in such a way as to remove the secular drift over time. The scale factor ℓ does not affect the equations of motion for bodies or light. However, it does alter the relationship between the rate of an atomic clock that records proper time on Earth (which is fixed at TT rate) and the rate of coordinate time in the barycentric frame.

The coordinate timescale defined in this fashion is termed *Barycentric Dynamic Time*, or TDB, and is the independent variable in all ephemerides used by the MPG. It is synonymous in this work with the term *ephemeris time*, except where otherwise specifically noted. It differs from TT by at most by 1.7 ms.

$$TCB = \ell\, TDB = (1 + L)\, TDB \qquad (2\text{-}5)$$

The value required to render no long-term secular variation between TT and TDB has been determined (see the Appendix to this chapter) to be $L = 1.550505 \times 10^{-8}$.

The MPG uses two conversion routines provided by NAIF to translate between TAI and TDB. These are `HPTA2E` (High Precision Time, Atomic to Ephemeris) and `HPTE2A` (High Precision Time, Ephemeris to Atomic). It also uses two routines that translate between UTC and TDB; these are `HPTU2E` (High Precision

Time, UTC to Ephemeris) and HPTE2U (High Precision Time, Ephemeris to UTC). Their usage is documented in the code commentary.

The time translation formula used in these utility functions was derived by JPL's Ted Moyer (see Moyer81). Moyer's derivation of the TDB-TAI relationship is lengthy and detailed, as it also analyzes the magnitudes of all elements of the approximation, including terms which were then omitted from the final result as being inconsequential, insofar as its use in the ODP was concerned. An abbreviated version of that method appears in the Appendix to this chapter for readers who may wish to have insight into the relativistic theory of the transformation. Those with further interest may consult the reference, which is now archived in electronic form. Those with less may skip the Appendix altogether.

2.10 The TDB-TAI Equation

Moyer's 1981 paper [Moyer81] derives two forms of the TDB-TAI formula. One is a numeric expression involving the clock location and properties of various mass centers (e.g., gravitational constant, velocity, and longitude of Jupiter, Saturn, and the Earth-Moon barycenter). This formula appears in the ESAA (2.222). The other is the vector form shown below.

The vector form of the solution gives $TDB - TAI$ with considerably less computation and slightly greater accuracy than does the expression given in the ESAA.

The vector form of solution, used in the NAIF high-precision time routines, is

$$\begin{aligned} TDB - TAI = \Delta T_A &+ \frac{2}{c^2}\left(\mathbf{r}_{SB} \cdot \dot{\mathbf{r}}_{SB}\right) + \frac{1}{c^2}\left(\mathbf{r}_{BE} \cdot \dot{\mathbf{r}}_{B}\right) + \frac{1}{c^2}\left(\mathbf{r}_{EA} \cdot \dot{\mathbf{r}}_{E}\right) + \\ &+ \frac{1}{c^2}\left(\mathbf{r}_{SB} \cdot \dot{\mathbf{r}}_{S}\right) + \frac{1}{c^2}\left(\frac{m_J}{m_S + m_J}\right)\left(\mathbf{r}_{SJ} \cdot \dot{\mathbf{r}}_{SJ}\right) + \\ &+ \frac{1}{c^2}\left(\frac{m_{Sa}}{m_S + m_{Sa}}\right)\left(\mathbf{r}_{SSa} \cdot \dot{\mathbf{r}}_{SSa}\right) \end{aligned} \tag{2-6}$$

in which $\Delta T_A = 32.184s$, $\mathbf{r}_{xy}$ denotes a vector from point x to point y, overdot denotes differentiation with respect to TDB, $m_b = G M_b$ is the gravitational constant of body b, and A, E, B, S, J, and Sa refer, respectively, to the positions of the atomic clock, the geocenter, the Earth-Moon barycenter, the Sun, Jupiter, and Saturn.

This equation is the same for all Earth-fixed clocks, whether on the geoid or not, as no restrictions are made as to clock location. However, evaluation of the equation does require the clock's location relative to the geocenter, and is therefore does not yield the same result for all DSSs. This is a natural result, whose first-order results are predicted by special relativity.

2.11 Evaluation of TDB Given TAI

Because the $TDB - TAI$ formula is implicitly based on TDB, then given a TDB value, the difference may be computed and TAI found via the relationship

$$TAI = TDB - (TDB - TAI) \tag{2-7}$$

However, when TAI is given, TDB must be found by iteration. The estimation of TDB_n by

$$TDB_n = (TDB - TAI)_{n-1} - 1 + TAI \tag{2-8}$$

fortunately converges very rapidly. The first two terms of the $TDB - TAI$ formula are the most significant, and provide the means for a good first estimate of TDB.

The convergence rate is illustrated by considering the difference between two succeeding estimates,

$$\begin{aligned} TDB_n - TDB_{n-1} &= (TDB - TAI)_{n-1} - (TDB - TAI)_{n-2} \\ &\approx \frac{d\Delta t}{dTDB}(TDB_{n-1} - TDB_{n-2}) \end{aligned} \tag{2-9}$$

The derivative factor has a value whose maximum magnitude is much smaller than unity (on the order of 10^{-9}). Each succeeding estimate is therefore much closer to the correct value than the preceding one.

This iterative method is used in the NAIF `HPTA2E` utility.

2.12 Universal Time (UT)

Prior to 1925, mean solar time was reckoned from noon in astronomical practice, and designated Greenwich Mean Time, or GMT. Beginning in 1925, a 12h discontinuity was introduced in order that mean solar time was reckoned from midnight, rather than noon. The *Nautical Almanac* used GMT for the new reckoning, whereas, the *American Almanac* used Greenwich Civil Time, or GCT, for it. This confusion in terminology was finally removed by dropping both designations and instituting Universal Time, or UT.

However, in the United Kingdom, GMT is sometimes still used. For civil timekeeping, it means UTC, and for navigation, it refers to UT1. Thus, GMT has two meanings that can differ by as much as 0.9s, and is not used in DSN subsystems.

UT is a measure of time that closely approximates the mean diurnal motion of the Sun. UTC, which is really an atomic time scale for civil timekeeping, is discussed separately a little late in this Chapter. Several other forms of UT exist and are used within the MPG, principally for determination of Earth's attitude at a given UTC instant.

The timescale designated as UT0 is determined directly from observations of the diurnal motions of the stars; it is slightly dependent on the place of observation. The timescale UT1 is obtained by correcting UT0 for the shift in longitude of an observing station caused by polar motion. Whenever the designation UT is used in this document, UT1 is implied. UT1 contains 41 short-period terms between 5 and 35 days that are caused by long-period solid Earth tides, designated as DUT1. When these are removed from UT1, the result is designated as regularized universal time, or UTR.

UT1 is defined in such a way that it can be directly related to mean sidereal time through a mathematical formula, appearing later in this Chapter. It thus does not refer to the motion of Earth, nor is it precisely related to the Sun's hour angle.

The apparent diurnal motion of the Sun involves both the nonuniform rotation of Earth and the motion of Earth in its orbit around the Sun. UT1 was not based on the hour angle of the Sun because such a system of time measurements could not be related precisely to sidereal time, and could not be determined by observations star transits and other such celestial measurements.

UT1 is counted from 0 hours at midnight, with unit of duration the *mean solar day*, defined to be as uniform as possible despite variations in the rotation of Earth. It is continuous (no leap seconds), but has a somewhat variable rate due Earth's non-uniform rotational period.

The International Earth Rotation Service (IERS) tabulates the difference between UT1 and UTC, $\Delta UT1 = UT1 - UTC$, as IERS Bulletin B, which is available via the Internet. However, the MPG receives this data from the Kalman Earth Orientation Filter (KEO) interface, along with other Earth Orientation Parameters (EOP). UT1 is then calculated as

$$UT1 = UTC + \Delta UT1 \tag{2-10}$$

2.13 Coordinated Universal Time (UTC)

Coordinated Universal Time (UTC) was established on January 1, 1972 at 0h as a measure of time that now serves as the basis for almost all civil timekeeping. It is commonly accessed within the United States via broadcasts by the NIST radio station WWV and other services.

As discussed above, it is related to Universal Time (UT), which is mean solar time, but UTC is really an atomic time. UTC uses the SI second as it fundamental unit and is adjusted infrequently in order to maintain the transit of the prime meridian at a time near noon. Since Earth's rotation is not uniform with respect to atomic time, a *leap second* is added or subtracted as necessary to prevent the error between UTC and mean solar time from exceeding 0.9s. Leap seconds are typically added at the end of December or June, but they can also appear (added or subtracted) at other designated times throughout the year.

$$UTC = TAI - (number of\ leap seconds) \tag{2-11}$$

Leap second adjustments affect the number of seconds per day and thus the number of seconds per year. The number of leap seconds incorporated into the January 1, 1972 initial UTC epoch was 10. Subsequently, the UTC timescale has marched backward relative to the TAI timescale exactly one second on each of a number of scheduled occasions.

Precisely when leap seconds occur, and in which direction the correction is made, is currently the responsibility of the IERS, which publishes periodic bulletins

(Bulletin C) available via the Internet. As specified in CCIR Report 517, a leap second is inserted following second 23:59:59 on the last day of June or December and becomes second 23:59:60 of that day. A leap second would be deleted by omitting second 23:59:59 on one of these days, although this has not yet happened, as of this writing.

On July 4, 2005, the EIRS announced the introduction of the 33rd leap second on December 31, 2005. The 32nd had been introduced on December 31, 1998.

The MPG converts UTC into TDB, and vice versa, through use of the NAIF special routines `HPTU2E` and `HPTE2U`, which require the presence of a file containing a time-tagged list of leap seconds, or, in SPICE parlance, a leap seconds kernel.

Earth is divided in to standard-time zones, and local times differ from UTC by an integral number of hours according to the particular time zone. Parts of Canada and Australia differ by integer-plus-half hours.

UTC times are referenced to the Zero meridian (Greenwich, England), which is often designated by a "Z" affixed to the time format, as 12:45:03Z. Z was thus designated as the international time zone for the prime meridian. It is sometimes thus referred to phonetically as "Zulu" time. The U.S. local time zones are Eastern ["R", "Romeo]; Central ["S", "Sierra"]; Mountain ["T", "Tango"]; Pacific ["U", "Uniform"]; Alaska ["V", "Victor"], and Hawaii ["W", "Whiskey"].

2.14 Sidereal Time (ST)

True sidereal time measures the Greenwich hour angle of the true equinox of date, measured westward from the true prime meridian of date (0°) about the true pole of date to the true vernal equinox of date. As such, then, sidereal time is a direct measure of Earth's diurnal rotation. The MPG ordinarily uses the Precision Earth Model (PEM) for determining DSS locations and SPICE utilities, such as `SPKEZ`, for spacecraft positions. However, it does require sidereal time to calculate the directions of radio sources. For this reason, a discussion is included here.

The period of time between two consecutive upper meridian transits of the equinox is a *sidereal day*. Since the rotation of Earth is subject to irregular forces, sidereal time is irregular with respect to atomic time. The practical determination of sidereal time from observations of radio sources and other means requires inclusion of Earth's precession, nutation, and polar motion effects.

2.15 Greenwich Mean Sidereal Time

.*Mean sidereal time* is referenced to the mean equinox of date, which is perturbed only by precession. *Greenwich Mean Sidereal Time* (GMST) is directly related to UT1 by a numerical formula. Since January 1, 1984, Greenwich Mean Sidereal Time (GMST), measured in seconds, at 0h UT1 on day d, measured in days past JD2451545.0, is given by the following formula appearing in the ESAA,

$$\begin{aligned} GMST_0^s &= 24110.54841 + 236.5553679087\,d + \\ &+ 6.9789\times 10^{-11}\,d^2 - 1.27\times 10^{-19} d^3 \end{aligned} \tag{2-12}$$

Note in this expression that Julian Dates begin at noon, so a given d at 0h has a fractional part of 0.5. Since the sidereal year is a full day longer than a solar year, each sidereal day is about $86400/365.25 = 236.55s$ longer than a UT day. Translated into units of degrees, this formula is

$$\begin{aligned} GMST_o^{\circ} &= 100.46061838 + 0.985647366286\,d + \\ &+ 2.9079\times 10^{-13}\,d^2 - 5.3\times 10^{-22} d^3 \end{aligned} \tag{2-13}$$

That is, the Greenwich hour angle of the equinox advances about one degree per day. The conversion of this expression into a general expression for mean sidereal time at any UT1 instant was done by Moyer (see Moyer2000, Eq. 5-173). His result, with d now including days plus fractional days past J2000, is

$$\begin{aligned} GMST^s &= 67310.54841 + 24.06570982442\,d \\ &\quad 6.9789\times 10^{-11} d^2 - 1.27\times 10^{-19}\,d^3 \end{aligned} \tag{2-14}$$

Expressed in angular measure, this is

$$\begin{aligned} GMST^{\circ} &= 280.460618375 + 360.9856473663 d + \\ &\quad 2.9079\times 10^{-13} d^2 - 5.3\times 10^{-22}\,d^3 \end{aligned} \tag{2-15}$$

2.16 Greenwich Apparent Sidereal Time

Apparent sidereal time is referenced to the true equinox (the intersection of the true equator of date with the ecliptic of date), and thus includes the effects of both precession and nutation. *Greenwich Apparent Sidereal Time* (GAST) is related to GMST via the *equation of the equinoxes* (before 1960, this effect was called "nutation in right ascension"). Nutation describes the motion of the true pole relative to the mean pole and may be resolved into components in longitude and obliquity. The longitude component, between the mean equinox and the true equinox of date, is denoted Δy , and the angle between the mean ecliptic of date and the true equator of date is the true obliquity, and denoted e . Both of these quantities are available via IERS bulletins, and to the MPG via the KEO Filter interface.

The difference between the true and mean right ascensions of a position on the true equator is the difference between apparent and mean sidereal time,

$$
\begin{aligned}
GAST &= GMST + EE \\
EE &= \Delta y \cos e + CT
\end{aligned}
\tag{2-16}
$$

where EE denotes the *equation of the equinoxes* and CT includes the "complementary terms" that were added to the "textbook" form of the equation of the equinoxes by IAU Resolution C7, Recommendation 3 (1994). The new formulation takes into account cross-terms between the various precession and nutation quantities, amounting to about 3 milliarcsec (83 microdeg). The transition from the old to the new model officially took place on February 27, 1997. These terms were added to compensate for irregularities in the UT1 timescale traceable to side effects of nutation and precession. By convention, the complementary terms were included in the equation of the equinoxes, rather than as apart of mean sidereal time.

2.17 Approximate Equation of the Equinox

The U. S. Naval Observatory (USNO) publishes approximate formulas for sidereal time, which may be accessed via the internet. Their 1978 formulas for GMST and EE, based on an epoch of 00:00:00 January 1, 1900, were incorporated into the NSS MP. An updated set is available that uses the J2000 epoch.

The expression for GMST in this set agrees with the one given above; the approximation for EE is

$$\begin{aligned} EE^\circ &= \Delta y^\circ \cos e \\ \Delta y^\circ &= -0.004785 \sin \Omega - 0.00036 \sin 2L \\ \Omega^\circ &= 125.04 - 0.052954d \\ L^\circ &= 280.47 + 0.98565d \\ e^\circ &= 23.4393 - 0.0000004d \end{aligned} \tag{2-17}$$

In this formulation, Ω is the longitude of the ascending node of the Moon's orbit and L is the mean longitude of the Sun. All expressions are given in degrees.

The cited maximum error resulting from the use of these formulas over the period 2000-2100 is 0.432 seconds, with an rms error of 0.15 seconds. At Earth's diurnal rate, the maximum error amounts to about 1.8 mdeg. Since this exceeds the MPG pointing error specification, it is unusable.

2.18 Local Apparent Sidereal Time

Local Apparent Sidereal Time (LAST) is the apparent right ascension of the local meridian. It can be obtained from the GAST by adding the meridian's east longitude.

2.19 Solar Time

Although solar time is not used in the MPG, it nonetheless has historical importance that mandates its inclusion in this Explanatory Supplement. *Apparent Solar Time* is the oldest timescale of all, being the measure of time defined by the actual diurnal motion of the Sun. It has been known since antiquity that the Sun's motion is not uniform, a fact that became well established after the invention of clocks, which tended to measure uniform intervals of time. Apparent Solar Time was the argument in *The National Almanac* and other national ephemerides until the early nineteenth century. *Mean Solar Time* was defined by the motion of an imaginary fiducial body (the *fictitious mean Sun*) that moved uniformly in the equatorial plane at a rate virtually equal to the mean rate of the true Sun's motion in the ecliptic. As clocks improved and came into extensive use at sea in the late eighteenth and early nineteenth centuries, apparent time was eventually superseded in civil use by Mean Solar Time. In the mid-nineteenth century, it became the argument in the national ephemerides.

At first, astronomers began counting time in hours past noon, so the Sun's meridian distance was indeed the time of day. When the time of day began to be reckoned at midnight, it was necessary to add 12 hours to time to determine the location of the fictitious mean Sun at that time.

The difference between apparent and mean solar times is called the *equation of time*. The principles for determining the equation of time extend back at least to the time of Ptolemy. At first, the equation of time was a way of determining mean solar time from apparent solar time. As clocks improved, it became a way of calculating apparent solar time from clock time, which kept, to the accuracy available at the time, mean solar time. These clocks were regulated by observations of sidereal time.

The underlying concept of mean solar time was that the rotation of Earth is uniform. However, in the first half of the twentieth century it became obvious that this assumption could no longer be deemed acceptable. To replace it, two new timescales came into being. Ephemeris Time (ET) was introduced to satisfy the desire for a uniform measure of time that would be the independent variable in the mathematical computation of ephemerides, and Universal Time (UT) came to designate a measure of Earth's rotation, as discussed in previous Sections of this Supplement.

The irregularities in apparent solar time are principally due to two effects. First, the motion of Earth in the ecliptic plane is not uniform, but is almost elliptical. The difference between the true anomaly and mean anomaly of Earth's orbit is an angle through which Earth must turn in order for the Sun to transit the local meridian. The time required for Earth to rotate by this angle is therefore one component of the equation of time.

The second effect is due to the inclination of Earth's pole to the ecliptic, which makes the Sun, as viewed at Earth, traverse above and below the equatorial plane. The projection of the Sun's path on the equatorial plane thus causes further distortions of the apparent motion.

Both effects are annual phenomena, and tend to produce a characteristic cyclic variation over time. A formula for the equation of time appears in the ESAA

(9.311). However, the algorithm given below agrees with this formula within a few seconds, and is more intuitively understood.

The procedure is to compute the true anomaly in the ecliptic, project this onto the equatorial plane, and subtract the mean motion in the equatorial plane, as follows: Calculate Earth's mean anomaly and obliquity, in radians (reduced, as necessary, to be in $0-2\pi$), using

$$\begin{aligned} M &= 6.2400407681 + 0.0172019698861 d \\ \varepsilon &= 0.409092959363 - 6.36892077921 \times 10^{-9} d \end{aligned} \tag{2-18}$$

where d is the number of UT days past the J2000 epoch. Next, invert Kepler's equation[2] to find the eccentric anomaly E

$$M = E - e \sin E \tag{2-19}$$

Then use Earth's orbital eccentricity $e = 0.01671$ to compute the current longitude (true anomaly)

$$\theta = \tan^{-1}\left(\sqrt{1-e^2}\,\sin E / (\cos E - e)\right) \tag{2-20}$$

Use the two-argument arc-tangent function, if available, to retain proper quadrant information.

Rotate the ellipse in the ecliptic clockwise about the z axis by the adjusted true anomaly, $\theta - \Omega$, using a value for perihelion elongation from winter solstice of $\Omega = -0.223796326795$,

$$\begin{aligned} x1 &= \cos(\theta - \Omega) \\ y1 &= \sin(\theta - \Omega) \\ z1 &= 0 \end{aligned} \tag{2-21}$$

Rotate the ecliptic plane counterclockwise about the y axis by the obliquity

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[2] Iteration of Kepler's equation converges rapidly, with $E_0 = M$, $E_i = M - e \sin E_{i-1}$, until $|E_{i+1} - E_i| < 10^{-4}$. This provides subsecond accuracy of conversion.

$$\begin{aligned} x2 &= \cos e \; x1 + \sin e \; z1 \\ y2 &= y1 \\ z2 &= -\sin e \; x1 + \cos e \; z1 \end{aligned} \tag{2-22}$$

Finally, rotate the equatorial plane counterclockwise about the z axis by the adjusted mean anomaly,

$$\begin{aligned} x3 &= \cos(M - \Omega)\, x2 + \sin(M - \Omega)\, y2 \\ y3 &= -\sin(M - \Omega)\, x2 + \cos(M - \Omega)\, y2 \\ z3 &= z2 \end{aligned} \tag{2-23}$$

The equation of time, in minutes, and declination of the Sun, in degrees, are

$$\begin{aligned} EOT &= -\frac{720}{p} \tan^{-1}(y3/x3) \\ d &= \frac{180}{p} \sin^{-1}(z3) \end{aligned} \tag{2-24}$$

The leading fractions are conversion factors for the cited units.

The resulting behavior plotted over time is shown in Figure 2-2. A plot of the equation of time versus declination, known as an *analemma*, is shown in Figure 2-3.

2.20 Time Representations and Standard Formats

Epochs and intervals have different representations to humans than they have in machine form. As mentioned earlier, SPICE and the MPG interpret times internally in the form of seconds past the J2000 epoch. Human readable contexts are usually expressed as strings of characters.

However, the variety of ways people have developed for expressing times in string formats is enormous. It is unlikely that any single software package can accommodate all of the custom time formats that have arisen in various computing contexts. However, SPICE and NAIF utilities correctly interpret most time formats used throughout the planetary science community. It supports ISO, UNIX, VMS, and MS-DOS formats, with epochs in both AD and BC eras, and with time zone specifications. These utilities transform from string to machine form and from machine form into strings. Consult the SPICE required reading document TIME.REQ for complete details on formats and translations.

Some of the more frequently human-readable formats are listed below. The ISO formats specify strict forms with required terms and exact punctuation. The SPICE and NAIF routines are much more general. In ISO formats a "T" is required to indicate the beginning of a time specification.

Year	YYYY
Year and Month	YYYY-MM
Complete Date	YYYY-MM-DD
Complete Date and Time	YYYY-MM-DDThh:mm:ss.ffTZD
Year and Day of Year	YYYY-DOY
Year and Day of Year, plus Time	YYYY-DOY DDThh:mm:ss.ffTZD

Times are generally assumed to be expressed in UTC, with a special UTC designator ("Z"). Local times in ISO formats require offset designators.

In this table, the representations are

YYYY	four-digit year
MM	two-digit month (01 is January, etc.)
hh	two-digit hour (00 through 23). No "am/pm" in ISO.
mm	two-digit minute (00 through 59)
ss	two-digit seconds (00 through 59)
f	one or more digits representing decimal fractions of a second
TZD	time zone designator (e.g., Z) or offset (+hh:mm or –hh:mm)

SPICE utilities recognize a wider range of more user-friendly formats that are not as strict as the ISO forms. For example, they also recognize Julian Dates, "am" or "pm", and U.S. local time zone designations (e.g., PDT), as well as timescale designations (e.g., UTC, TAI, TDB, TDT).

2.21 Spacecraft Atomic Time

Given the spacecraft identifier and the ephemeris time of interest, the NAIF-supplied function `HPTDAT` produces the derivative of Spacecraft Atomic Time (TAS[3]) with respect to ephemeris time in the form $D_t = \dot{\ }(t) - 1$, where t is a TDB time and (t) is its equivalent atomic time aboard the given spacecraft.

At one stage of its development, the MPG computed spacecraft atomic time in order to compute one-way light time values. It therefore generated a polynomial profile that approximated $\Delta \ (t) = \ (t) - t$ within a prespecified error value. This data type was deemed unneeded for a number of reasons, and was later omitted from the MPG design.

The reasons for not using the estimated TAS were that (1) one-way light time data products are used differentially, so that TAS disappears in the differences,

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[3] The word order here matches that of TDB and TAI, even though it does not represent a French word order. "Spacecraft" in French would become "Astronef" or some other suitable term.

and (2) the TAS profile contained a constant of integration that could not be evaluated. Nevertheless, the means that were used in the earlier design are included here for historical purposes.

The MPG produced the TAS polynomial profile by segmenting the time interval of interest (normally a pass) into intervals (t_i, t_{i+1}), sampling D_t at five optimally spaced points over each interval, and integrating the results to form a profile of polynomials of degree 5. Each polynomial in the profile was expressed in the standard MPG Everett form. The samples values extracted were

$$\mathbf{y} = (\;{}_{i}, D_t(t_i), D_t(t_i + 33\,h/181), D_t(t_i + h\,/2), D_t(t_i + 148h\,/181), D_t\,(t_{i+1}))^T \quad (2\text{-}25)$$

Here h is the interval length and t_i is time at the beginning of the i^{th} interval. The method of generating the polynomial from sample values is treated in the chapter on Interpolation in this Supplement.

In this formulation, the value $\;{}_{i}$ was recognized as a constant of integration, to be evaluated. Given this value at the beginning of an interval, the subsequent endpoint value $\;{}_{i+1}$ will be determined by the interval length and linear combinations of sample values. To make the profile continuous at interval boundaries, the beginning value of the next interval becomes the final value of the previous one throughout the profile.

Only the $\;{}_{0} = \;(t_0)$ of the initial interval thus remains unspecified. This initial offset between spacecraft atomic time and ephemeris time is unknown, and was therefore set to zero in the earlier MPG development. Since users of predictions involving TAS were generally insensitive to the true difference, this choice is of no known consequence.

Appendix A

Derivation of TDB-TAI

This Appendix presents an overview of the method found in Moyer's [Moyer81] derivation of the relationship between TDB and TAI. Computation of the scaling factor L is mentioned in the reference, but not directly computed. The approximate computation of the scaling factor that appears below is due to the author.

Computing the Scaling Factor

If the scaled time (TDB) is denoted by T , where

$$t = \ell T \tag{2-26}$$

then the relationship between proper time and TDB is

$$\frac{dt^2}{dT^2} = \frac{ds^2}{c^2 dT^2} = \ell^2 \left[\left(1 - \frac{2U}{c^2}\right) - \left(1 + \frac{2U}{c^2}\right)\frac{v^2}{c^2} \right] \tag{2-27}$$

in which v is the clock's velocity relative to the solar system barycenter, translated into the barycentric frame.

If it is desired that no periodic terms in this equation appear, ℓ can be set to the value

$$\ell = \left\langle \left[\left(1 - \frac{2U}{c^2}\right) - \left(1 + \frac{2U}{c^2}\right)\frac{v^2}{c^2} \right]^{-1/2} \right\rangle \tag{2-28}$$

in which the $\langle\ \rangle$ operation removes short-term variations. A one-term Taylor expansion of the right hand side above in $1/c^2$ yields the evaluation

$$\ell = 1 + \frac{\langle U + v^2/2 \rangle}{c^2} \tag{2-29}$$

It is convenient to define L as the small departure from unity in this factor,

$$L = \frac{\langle U + v^2/2 \rangle}{c^2} \tag{2-30}$$

It remains only to evaluate the secular parts of U and v^2.

Approximating the Scaling Factor

An approximation method is useful to illustrate the method actually used. Suppose that the clock path is represented as a circular orbit at distance a about a given center, and let the path of a given gravitational influence be represented also by a circular orbit at distance b about the same center. Let $d = \max(a, b)$ and $r = \min(a, b)/d$. Then the distance between the gravity source and clock may be computed using vector algebra, which will result in an algebraic function involving trigonometric variations over time. Using an algebraic computational tool, such as *Mathematica*, the inverse distance function may be expanded in a Taylor series in r to determine the constant and periodic terms. The secular portion of the Newtonian potential for that gravity source is then found to be

$$\langle U_i \rangle = \frac{m}{d}\left(1 + \frac{r^2}{4} + \frac{9r^4}{64} + \cdots\right) \tag{2-31}$$

where $m = G\,M$ is the gravitational constant of the source. This expression may be applied to the Sun ($a = AU, b = 0$), Moon ($a = r_{Earth}, b = dist_{Moon}$), Earth ($a = r_{Earth}, b = 0$), and each of the planets ($a = AU, b = dist_{planet}$), to provide the approximate contributions of these bodies to the scale factor.

Similarly, the velocity may be represented a constant vector whose magnitude is the mean geocentric velocity about the barycenter added to a rotating vector whose magnitude is the mean diurnal clock velocity. The result is

$$v^2 \approx \left(\frac{2p\ AU}{365.25\ days}\right)^2 + \left(\frac{2p\ r_{Earth}}{1\ day}\right)^2 \tag{2-32}$$

The composite result is that $L \approx 1.55059 \times 10^{-8}$. The value published in the ESAA is $L = 1.550505 \times 10^{-8}$, and that used in the NSS MP was $L = 1.5505204 \times 10^{-8}$. The approximation has omitted eccentricities in orbits and the effects of considering the Earth-Moon motion about their barycenter, which are included in the actual computation. Nonetheless, the agreement is pretty good.

The rate difference between TDB and TCB thus turns out to be about 48.93 s/century. The rate difference between TDB computations using the ESAA rate and the one approximated above is only about 3 ms/century, and using the ESAA and NSS MP rates is only about 0.5 ms/century.

Computing the Periodic Part

In order to compute TDB, it is first necessary to determine the differential equation relating the rates of the two timescales. This may be done by square-rooting the metric equation. Then the right-hand side may be expanded in a Taylor series in $1/c^2$, with terms of order higher than $1/c^2$ discarded. The result is

$$\frac{dt}{dT} = 1 - \frac{U + v^2/2}{c^2} + L \tag{2-33}$$

It then remains only to integrate this differential equation. In practice, TDB (T above) is determined from TT (t above) by means of appropriate mathematical approximations. Secular terms disappear due to the presence of the L term. The desired form of solution is the time difference

$$T - t = \Delta T_0 + \int_{T_0}^{T} \left\{ \frac{U + v^2/2}{c^2} \right\} dT \tag{2-34}$$

where $\{Q\} = Q - L$ retains only the periodic parts of Q.

In 1981, JPL's Ted Moyer (Moyer 81) published two forms of solution. One had appeared in a JPL Technical Report several years earlier, in 1971. This is essentially the one that appears in the ESAA, and is the one that is used by the NSS MP. The 1981 article also developed a vector form of the solution, which is the one used in the MPG. Moyer's article cites others who have derived similar formulas. In fact, several formulas appear have greater accuracies (errors as low as 1 ns, but nominally –131 to 64 ns) than the Moyer forms, but they all contain considerably more (127 to 1637) terms. The Moyer solutions are sufficiently accurate (within about 4 μs) for MPG applications, and the vector form is the one used in the MPG and ODP.

Moyer's derivation of the TDB-TT formula is lengthy and detailed, as it also analyzes the magnitudes of all elements of the approximation, including terms which were then omitted from the final result as being inconsequential, insofar as its use in the ODP was concerned. An abbreviated version of that method will be presented here for readers who may wish to have insight into the relativistic theory of the transformation. Those with further interest may consult the reference, which is now archived in electronic form. Those with less may skip the remainder of this subsection.

Since only periodic terms remain in the solution, only these will be retained in the analysis as it progresses. The constant terms, of course, accumulate into the L parameter, and provide its more accurate determination.

The following notation will be used to designate vector positions and velocities: $\mathbf{r}_a$ denotes the position of the entity a with respect to the solar system barycenter, and $\dot{\mathbf{r}}_a$ denotes its derivative with respect to coordinate time (velocity); $\mathbf{r}_{ab}$ denotes the vector $\mathbf{r}_b - \mathbf{r}_a$ and $\dot{\mathbf{r}}_{ab}$ is its time derivative. Entities of interest are the Sun (S), Moon (M), Earth (E), the Earth-Moon barycenter (B), Jupiter (J), Saturn (Sa), and the atomic clock (A) that is recording TT. The other planets are indexed by their numerical order outward from the Sun.

The integrand contains the quantity

$$U_A + v_A^2 / 2 \tag{2-35}$$

which is the sum of the potential and kinetic energies per unit mass at the clock location. The first step is to express each of these terms in geocentric terms,

$$\begin{aligned} U_A &= (U_A - U_E) + U_E = U_E + \nabla U_E \cdot \mathbf{r}_{EA} + \cdots \\ v_A^2 / 2 &= \dot{\mathbf{r}}_A \cdot \dot{\mathbf{r}}_A / 2 = (\dot{\mathbf{r}}_{EA} + \dot{\mathbf{r}}_E) \cdot (\dot{\mathbf{r}}_{EA} + \dot{\mathbf{r}}_E) / 2 = v_{EA}^2 / 2 + \dot{\mathbf{r}}_{EA} \cdot \dot{\mathbf{r}}_A + v_E^2 / 2 \end{aligned} \tag{2-36}$$

The potential U_E is evaluated at the geocenter, and thus excludes Earth's gravitational effects.

In the second of these equations, v_{EA}^2 is the square of the geocentric velocity of the clock, which is constant if the effects of solid earth tides, polar motion, and nutation are ignored. It therefore may be dropped from further consideration. In the first equation, the remaining terms in the Taylor expansion of the potential are omitted as inconsequential. Further, Newton's law can be used to estimate the potential gradient term within the required order of $1/c^2$ terms

$$\nabla U_E \approx \ddot{\mathbf{r}}_E \tag{2-37}$$

so that

$$\nabla U_E \cdot \mathbf{r}_{EA} = \ddot{\mathbf{r}}_E \cdot \mathbf{r}_{EA} = \frac{d(\dot{\mathbf{r}}_E \cdot \mathbf{r}_{EA})}{dT} - \dot{\mathbf{r}}_E \cdot \dot{\mathbf{r}}_{EA} \tag{2-38}$$

Notice that the negative term in this equation cancels a term in the earlier velocity equation. The differential time equation time periodic terms are now

$$\frac{d(T-t)}{dT} = \frac{d\,\Delta t}{dT} = \frac{U_E + v_E^2/2}{c^2} + \frac{d(\dot{\mathbf{r}}_E \cdot \mathbf{r}_{EA})}{dT} \tag{2-39}$$

Moreover, the derivative term on the right-hand side can be integrated directly, leaving only the first term to be further manipulated.

The same kind of action can be applied to the variation about the Earth-Moon barycenter, in which $U_E = (U_E - U_B) + U_B$ and $\mathbf{r}_E = \mathbf{r}_{EB} + \mathbf{r}_B$. The U_B term is evaluated at the Earth-Moon barycenter, and excludes terms due to Earth and Moon. The result is

$$\frac{d\,\Delta t}{dT} = \frac{U_B + v_B^2/2}{c^2} + \frac{d(\dot{\mathbf{r}}_E \cdot \mathbf{r}_{EA})}{dT} + \frac{d(\dot{\mathbf{r}}_B \cdot \mathbf{r}_{BE})}{dT} \tag{2-40}$$

Again, only the first term on the right requires further manipulation.

The remaining potential term,

$$U_B = U_{B,S} + \sum_i U_{B,i} \tag{2-41}$$

is the sum of the gravitational potential at B due to S and due to each of the other 8 planets. However, only the contributions due to Jupiter and Saturn were found to be significant, so only these terms are assumed in what follows.

$$U_{B,S} = \frac{m_S}{r_{SB}}$$
$$U_{B,i} = \frac{m_i}{r_{Bi}} = \frac{m_i}{|\mathbf{r}_{SB} - \mathbf{r}_{Si}|} = \frac{m_i}{r_{Si}}\left(1 + r_i \cos(l_i) + \frac{3}{4} r_i^3 \cos(2l_i)\right) \tag{2-42}$$

The quantity m_i is the gravitational constant of the planet i, or, when it references the Earth-Moon barycenter, to the combined gravitational constants of Earth and Moon. The r_i parameter is the same as that defined earlier, the periodic terms of the earlier Taylor expansion are now retained, and l_i is the longitudinal angle between i and B as viewed at S. Moyer's work determined that only the first two terms in the expansion were found to be needed for the accuracy required.

$$U_{Bi} = \frac{m_i}{r_{Si}} + \frac{m_i}{r_{Si}^2} r_{SB} \cos(l_i) \tag{2-43}$$

As for the velocity term, since $\mathbf{r}_B = \mathbf{r}_S + \mathbf{r}_{SB}$, v_B^2 becomes

$$v_B^2 = v_S^2 + v_{SB}^2 + 2\dot{\mathbf{r}}_S \cdot \dot{\mathbf{r}}_{SB} \approx v_{SB}^2 + 2\dot{\mathbf{r}}_S \cdot \dot{\mathbf{r}}_{SB}$$
$$\approx v_{SB}^2 + 2\left(\frac{d\,\dot{\mathbf{r}}_S \cdot \mathbf{r}_{SB}}{dT} - \ddot{\mathbf{r}}_S \cdot \mathbf{r}_{SB}\right) \tag{2-44}$$

The approximation assumes that the squared velocity of the Sun about the solar system barycenter has negligible periodic terms. As earlier, the acceleration term may be replaced by its Newtonian potential within the accuracy limits imposed,

$$\ddot{\mathbf{r}}_S = -\sum_i \frac{m_i}{r_{Si}^3} \mathbf{r}_{Si} \tag{2-45}$$

The sum extends over the Earth-Moon barycenter and retained planets. The dot product $\ddot{\mathbf{r}}_S \cdot \mathbf{r}_{SB}$ is then

$$\ddot{\mathbf{r}}_S \cdot \mathbf{r}_{SB} = -\sum_i \frac{m_i}{r_{Si}^3} \mathbf{r}_{Si} \cdot \mathbf{r}_{SB} = -\sum_i \frac{m_i}{r_{Si}^2} r_{SB} \cos(l_i) \tag{2-46}$$

These terms cancel with corresponding terms in the expansion of U_B. The resulting time differential equation is now

$$\frac{d\Delta t}{dT} = \frac{1}{c^2}\left\{\frac{m_S}{r_{SB}} + \frac{v_{SB}^2}{2} + \frac{m_J}{r_{SJ}} + \frac{m_{Sa}}{r_{SSa}} + \frac{d\,\dot{\mathbf{r}}_S \cdot \mathbf{r}_{SB}}{dT} + \frac{d(\dot{\mathbf{r}}_E \cdot \mathbf{r}_{EA})}{dT} + \frac{d(\dot{\mathbf{r}}_B \cdot \mathbf{r}_{BE})}{dT}\right\} \tag{2-47}$$

The remaining steps presume that the Earth-Moon barycenter, Jupiter, and Saturn are in heliocentric elliptical orbits. The squared-velocity term is given by *the vis-viva* equation,

$$v_B^2 = (m_S + m_E + m_M)\left(\frac{2}{r_{SB}} - \frac{1}{a_{SB}}\right) \tag{2-48}$$

in which a_{SB} is the ellipse semimajor axis. The Earth-Moon barycentric terms are then

$$\begin{aligned}\frac{m_S}{r_{SB}} + \frac{v_{SB}^2}{2} &= \frac{2m_S + m_E + m_M}{r_{SB}} - \frac{m_S + m_E + m_M}{2\,a_{SB}} \\ &\approx m_S\left(\frac{2}{r_{SB}} - \frac{1}{2\,a_{SB}}\right)\end{aligned} \tag{2-49}$$

in which the final terms contain no periodic elements, and may thus be dropped from consideration.

Also for an elliptic orbit, the inverse of the radial distance is a function of the eccentric anomaly E (not to be confused with the E used elsewhere to denote the position of Earth) and the ellipse eccentricity e, given by

$$\frac{1}{r_{SB}} = \frac{1}{a_{SB}} + \frac{e}{r_{SB}}\cos(E) \tag{2-50}$$

Only the latter term contains periodic elements, so the first may be omitted.

Further, the derivative of the eccentric anomaly with respect to coordinate time is

$$\dot{E} = \frac{1}{r_{SB}}\sqrt{\frac{m_S + m_E + m_M}{a}} \approx \frac{1}{r_{SB}}\sqrt{\frac{m_S}{a}} \tag{2-51}$$

The ratio of the left-hand side of this equation to the right-hand side is unity. Multiplication of the expression for the periodic elements of the Earth-Moon barycentric terms by this form of unity gives

$$\left\{\frac{m_S}{r_{SB}}+\frac{v_{SB}^2}{2}\right\}=\frac{2m_S\,e}{r_{SB}}\left(r_{SB}\sqrt{\frac{a_{SB}}{m_S}}\right)\cos(E)\,\dot{E}$$
$$=2e\sqrt{m_S a_{SB}}\,\cos(E)\,\dot{E} \tag{2-52}$$
$$=\frac{d}{dT}\left(2e\sqrt{m_S a_{SB}}\,\sin(E)\right)$$

Another property of elliptic orbits is that

$$\mathbf{r}_{SB}\cdot\dot{\mathbf{r}}_{SB}=e\sqrt{m_S a_{SB}}\,\sin(E) \tag{2-53}$$

As a consequence, then, an integrable form is obtained,

$$\left\{\frac{m_S}{r_{SB}}+\frac{v_{SB}^2}{2}\right\}=\frac{d}{dT}\left(2\,\mathbf{r}_{SB}\cdot\dot{\mathbf{r}}_{SB}\right) \tag{2-54}$$

A similar treatment of Jupiter and Saturn contributions provides

$$\left\{\frac{m_i}{r_{S\,i}}\right\}=\frac{d}{dT}\left(\frac{m_i}{m_S+m_i}\,\mathbf{r}_i\cdot\dot{\mathbf{r}}_i\right) \tag{2-55}$$

The TDB-TAI Equation

Since all terms in the time differential equation are now integrable, the final result is at hand. Integration adds a constant value ΔT_0 to the integrated periodic terms, which may be chosen to make $T-t$ have an agreed-upon value at a given epoch. Since TT and TAI are defined so as to differ only by a constant, the integrated equation can also express the difference $TDB-TAI$ by appropriate choice of the integration constant.

The 16th General Assembly of the IAU adopted the value $\Delta T_A = 32.184$ s. The final expression for the time differential is

$$
\begin{aligned}
TDB - TAI = {} & \Delta T_A + \frac{2}{c^2}\left(\mathbf{r}_{SB} \cdot \dot{\mathbf{r}}_{SB}\right) + \frac{1}{c^2}\left(\mathbf{r}_{BE} \cdot \dot{\mathbf{r}}_B\right) + \frac{1}{c^2}\left(\mathbf{r}_{EA} \cdot \dot{\mathbf{r}}_E\right) + \\
& + \frac{1}{c^2}\left(\mathbf{r}_{SB} \cdot \dot{\mathbf{r}}_S\right) + \frac{1}{c^2}\left(\frac{m_J}{m_S + m_J}\right)\left(\mathbf{r}_{SJ} \cdot \dot{\mathbf{r}}_{SJ}\right) + \\
& + \frac{1}{c^2}\left(\frac{m_{Sa}}{m_S + m_{Sa}}\right)\left(\mathbf{r}_{SSa} \cdot \dot{\mathbf{r}}_{SSa}\right)
\end{aligned}
\qquad (2\text{-}56)
$$

Moreover, this equation is the same for all Earth-fixed clocks, whether on the geoid or not, as no restrictions were made as to clock location.

This vector form of the solution gives $TDB - TAI$ with considerably less computation and slightly greater accuracy than does the expression given in the ESAA, which is a function of time and the topocentric coordinates of the atomic clock.

References

[Moyer1971] Moyer, T. D. (1971), "Mathematical Formulation of the Double Precision Orbit Determination Program,", Technical Report 32-1527, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, May 15, 1971.

[Moyer2000] Moyer, T. D. (2000), *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation*, JPL Publication 00-7, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, October 2000.

[Seidelmann1992] Seidelmann, K. P. (ed) (1992), *Explanatory Supplement to the Astronomical Almanac*, University Science Books, Mill Valley, CA,1992.

[NAIF1997] TIME.REQ (1997), SPICE Toolkit, Navigation and Ancillary Information Facility, Jet Propulsion Laboratory, Pasadena, CA, July 1997.

[Taus1998-1] Tausworthe, R. C., "Translation of Polynomials to and from Everett Form," *Mathematica* study `Everett-Horner.nb`, MPG Archive, NPP Release Directory, January, 1998.

[Taus1998] Tausworthe, R. C., and Walther, J. Y., "Estimation of Spacecraft Atomic Time from its Derivative Using the Error Ridge Method," *Mathematica* study `TAS-Ridge-Errors.nb`, MPG Archive, NPP Release Directory, July, 1998.