Chapter 4

Light Time and Frequency Prediction

The time required for a photon to travel from point T, the point of transmission, to point R, the point of reception, is called the light transit time between T and R, or more succinctly, the *light time*. The MPG calculates a number of types of light time predictions, and also generates various types of frequency predictions, which are related to the light time derivative. Inasmuch as they are intimately related, the two are treated together in this chapter.

Specification of a spacecraft's parameters and configurations to be used by the MPG is found in the project interface Prediction Parameters (PP) file. The name of the PP file is specified in the MPG Context File.

The MPG is then required to label all of its spacecraft predictions with the "correct" terminology, so the design relies on the user-supplied information to define the "correct" mappings among terms appearing in the file and the numeric values to which they apply. All subsequent calculations and labeling in the MPG flow from these values and mappings.

4.1 Conventions

The Deep Space Network (DSN) transmits carrier, commands, and ranging modulation to spacecraft and receives carrier, telemetry data, and returned ranging modulation at deep space tracking stations (DSSs), which are located within Deep Space Communication Complexes (DSCCs) that are located at strategic positions over the world. It is also engaged in radio science (RS) and very long baseline interferometry (VLBI) studies in which the deep space stations play a leading role.

The DSN is a part of a larger organization, the Interplanetary Network (IPN), which includes spacecraft and landers, and, eventually, will also include stations in orbit about or landed on other bodies in the solar system. Currently, the MPG is tasked only with producing predictions of events and observations that are made at deep space stations.

In particular, this chapter derives the formulas that describe the relationships among the transmission times and frequencies at a DSS, the reception times and frequencies at a spacecraft, the subsequent transmission times and frequencies at the spacecraft, and the succeeding reception times and frequencies at a DSS. The transmitting and receiving DSSs need not be the same. The configurations of interest are given descriptive names for reference.

"One-way" configurations are designations of uplink and downlink transmissions as separate entities. A "two-way" coherent geometry refers to uplink and downlink transmissions originated and received at a single given DSS during a time when the spacecraft is phase locked to the uplink carrier and retransmits its downlink in phase coherence with the uplink. A "three-way" coherent configuration is the same as two-way except that the transmitting DSS and the receiving DSS are different. Two- and three-way non-coherent geometries refer to links in which the spacecraft transmitted downlink is not derived from the uplink carrier.

Times and frequencies at DSS transmitter, spacecraft, and DSS receiver are tagged 1, 2, and 3, respectively. The subscripts "R" for "received" and "T" for "transmitted" are also used when needed or sometimes for emphasis.

Time epochs reckoned in an atomic time scale will here be denoted by τ (Greek tau). Those reckoned in ephemeris time will be denoted by a roman *t*. The speed of light in a vacuum is denoted by c .

The MPG's legacy system, the NSS MP, designated the source of proper time in a particular configuration by terminology that has also been carried forth into the MPG. A prediction is said to be *observed* if the data type is referenced to τ_3 , and *command* if referenced to τ_1 . Thus, a one-way uplink light time is called the command light time, and a one-way downlink light time is called the observed[1] light time. When a round-trip light time is referenced to τ_1 it is called the com-

[1] This nomenclature belies the relativistic usage of the term observable. This is discussed later in the chapter.

mand round-trip light time, and when referenced to τ_3, it is called the observed round-trip light time.

4.2 Ephemeris and Atomic Timescales

This section is a recap of information presented in foregoing chapters of this Supplement. It is provided here for review purposes.

Event times and frequencies at the DSSs and spacecraft are governed by the locations and motions of these entities over time. The positions and motions of the relevant entities are found in ephemerides that are supplied to the MPG, wherein quantities are given relative to a specified origin, coordinate frame, and timescale. The time stamp represents the independent variable of the motion of the bodies with respect to the given coordinate frame. It is not an observed time, but is strictly the value which corresponds to the geometrical coordinates of the ephemeris entity. It is thus sometimes referred to as *coordinate time*.

Observed times, however, require real clocks. Clocks at different locations appear to run at different rates with respect to any given clock due to relativistic effects. A fictitious clock located at the solar system barycenter (SSB) in an inertial frame is said to measure *barycentric coordinate time* or TCB (acronym in French word order). An observer located at the SSB would perceive an atomic clock on the Earth or on a spacecraft as a moving entity in a gravitational field, whose rate is different than its own.

In 1976 the International Astronomical Union (IAU) adopted *International Atomic Time* (or TAI, in French word order) as the standard independent variable of apparent geocentric ephemerides. Other time scales are then related to TAI through the relativistic equations relating to the reference frame and coordinate system in that frame. The unit of TAI is the International Second (SI) which is defined as a certain number of oscillations of an atomic standard clock situated on the geoid.

But TCB and TAI appear to run at diverging rates due to relativistic influences. So the IAU adopted another timescale, called *barycentric dynamic time*, or TDB, was defined so as not to diverge. The difference between TDB and TAI is thus made up of periodic terms that relate to the motion of the TAI clock relative to the SSB and the positions of bodies of significant gravitational influence in the solar system. The significant gravitational potential terms are those due to the Sun, Moon, and each of the planets (including Earth).

Traditionally, the term *ephemeris time* was used to denote the coordinate time of the solar system. The term is now somewhat passé, in that TDB is the preferred timescale among physicists, astronomers, and JPL Navigation. However, the term continues to be used in the MPG and by JPL Navigation for historic reasons and informal discussions. When used, it always, except when specifically noted otherwise, refers to TDB.

Civil timekeeping uses the Coordinated Universal Time (UTC) time scale, which differs from TAI by the current number of leap seconds. It differs from mean solar time by less than 0.9s, as maintained by an occasional introduction of a leap second increment.

In order to predict an observance from information given in an ephemeris, it is necessary to be able to translate between the time appearing on the observer's clock and that appearing in the ephemeris. The translation is different at each DSS because they all have different gravitational potentials and they all have different velocities with respect to the SSB. The MPG uses two conversion routines provided by NAIF to translate between TAI and TDB. These are `HPTA2E` (High Precision Time, Atomic to Ephemeris), which returns $\Delta\tau = t - \tau$, and `HPTE2A` (High Precision Time, Ephemeris to Atomic), which returns $\Delta t = \tau - t = -\Delta\tau$. The timescale in this notation is the input timescale. In addition to returning the converted times, these subroutines also return values related to the derivatives of TAI with respect to TDB and vice versa. `HPTA2E` returns the relativistic differential $D_\tau = dt \,/\, d\tau - 1$, and `HPTE2A` returns $D_t = d\tau \,/\, dt - 1$. The subscript in each case identifies the input timescale. Besides these utilities that translate TAI and TDB at Earth stations, there is also the utility `HPTADR` (High Precision Time, Atomic Derivative) that returns D_t for all locations having a NAIF identifier. Their usage is documented in the code commentary of these functions. The relativistic differentials are related by

$$\begin{aligned} D_\tau &= -\frac{D_t}{D_t + 1} \\ D_t &= -\frac{D_\tau}{D_\tau + 1} \end{aligned} \tag{4-1}$$

An earlier chapter established the differential relationship between ephemeris times and atomic times, equivalent to

$$D_t = -\frac{v^2}{2c^2} - \frac{U}{c^2} + L \tag{4-2}$$

Here U is the sum of the Newtonian gravitational potentials of the ensemble of masses evaluated at the position of the clock, v is the velocity of the clock with respect to the SSB, and L represents the value introduced so as to make the difference between atomic time and ephemeris time have only periodic terms. For TAI on the geoid, it is $L = 1.550520 \times 10^{-8}$.

Spacecraft Atomic Time, referred to here as TAS, follows the same differential equation given above, but in which the gravitational field at the spacecraft and the spacecraft's barycentric velocity are applied. The same value of L is used, under the argument[2] that the spacecraft clock, while still on Earth, was synchronized to TAI, so this L is a characteristic of its clock, now in space.

However, there is no NAIF utility that provides a general integrated solution for TAS[3] itself. Instead, the MPG generates a polynomial profile of D_t over the span of time of interest, and integrates this profile to produce the polynomial profile of $\Delta\tau_2 = (\tau_2 - t_2) + a_0$ as a function of t_2. Since the actual time difference is unknown, the constant of integration a_0 is set to a value such that ET and TAS are the same at the beginning of the profile,

$$a_0 = -(\tau_{2,bgn} - t_{2,bgn}) \tag{4-3}$$

As a result of this convention, it will be seen later that one-way light time predictions may differ by a constant amount from the unknown actual values. However, such predictions will be consistent with respect to all Earth stations during the profiled time period. Light time differences are immune to this offset.

If, in the future, the MPG eventually is extended to include observing stations elsewhere in the solar system, high precision time routines will require extension to accommodate the gravity sources of influence at their atomic clocks.

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[2] See Moyer2000, Section 11.4.

[3] Solutions do exist for Earth satellites, and, by extension, to satellites around other planets. See Moyer2000, Section 7.

4.3 Light Time Computation

The DSN is in the business of communicating with and tracking various space objects, such as spacecraft, solar system bodies, and radio sources. Predictions of light times of several sorts are required for operating the network and for gathering navigation and scientific information. As will be seen a little later, it is also required for predicting frequencies. This section discusses the basic kinds of light time predictions that are made and the mathematical bases for each.

4.3.1 Light Time Definition

The pertinent calculation of this section might seem to answer the simple question, “How long does it take for a photon to go from point T to point R?” But the answer to this question, as required of the MPG, is not simple at all. There are several components of the delay itself, such as distance, gravitational effects, atmospheric effects, and electronic pathways at both ends of the transmission. There is also a philosophical question of what is meant by “How long does it take…”.

Normally, relativity is played as an observer-event game. One observer compares what is seen or measured from one perspective with what is seen or measured from another viewpoint. Events that appear to be simultaneous in one frame may not appear to occur simultaneously in the other. Simultaneity is relative, and one generally deals with this fact by not trying to compare event times across frames. Almost every puzzle and apparent paradox in the theory of relativity hinges on some misconception about simultaneity.

One is therefore tempted to calculate “observed light time” in terms of proper time at the observer. If the observer is the receiver, then the epoch of reception is measured on the receiver’s clock, and the time of departure is reckoned as the ephemeris time of departure translated to the receiver’s clock. The difference between these two proper times is the light time as observed in the receiver frame of reference. An analogous light time is observed in the transmitter frame. These two time spans are not generally the same. That’s relative simultaneity for you!

However, this definition of light time is *not* the one used in MPG predictions. The MPG is not trying to predict what either frame might observe. Rather, the MPG definition of light time $L\tau$ is the difference between the temporal states of the transmitter and the receiver, as reckoned by their respective clocks. It is the time difference between transmitted and received atomic clocks. This definition provides the means for predicting when an event initiated in one frame may be expected to be culminated in another.

$$L\tau = \tau_R - \tau_T \tag{4-4}$$

4.3.2 Components of Light Time

Light time may be written in a form that displays times observed in each of the frames,

$$\begin{aligned} L\tau &= (\tau_R - t_R) + (t_R - t_T) + (t_T - \tau_T) \\ &= -\Delta\tau_R + LT - \Delta t_T \\ &= \Delta t_R + LT + \Delta\tau_T \end{aligned} \tag{4-5}$$

In this formulation, $LT = t_R - t_T$ represents the *ephemeris light time*, and the remaining terms are translations between SSB and atomic clocks, covered earlier. The two forms shown above display backward and forward solutions of the light time equation, discussed subsequently.

The state vectors of the T and R points relative to the SSB at the instants t_T of emission and t_R of reception may be determined as coordinates and velocities found in ephemerides corresponding to TDB time tags. The method by which these times are determined is discussed later in this chapter

Transmissions across the solar system between spacecraft and DSS are subject to temporal distortions due to gravitational effects along the path of travel. Light time is lengthened due to the slowing effect of gravity, and also due to the slightly longer path traveled due to the curvature of space by gravity. Moyer (see Moyer2000) derives an expression below for the additional light time incurred by each gravitational source of significance.

$$\Delta Lt(t_T,t_R)=\sum_{i=1}^{B}\frac{2\mu_i}{c^3}\ln\left(\frac{|\mathbf{p}_T^i(t_T)|+|\mathbf{p}_R^i(t_R)|+|\mathbf{p}_T^i(t_T)-\mathbf{p}_R^i(t_R)|}{|\mathbf{p}_T^i(t_T)|+|\mathbf{p}_R^i(t_R)|-|\mathbf{p}_T^i(t_T)-\mathbf{p}_R^i(t_R)|}\right) \quad (4\text{-}6)$$

where μ_i is the gravitational constant of the body of influence, B is the number of such bodies, and $\mathbf{p}_X^i$ denotes the position of point X relative to the point i, in this case a gravity source as obtained from ephemerides[4] with t_X is the coordinate time of the point's position.

There is also atmospheric delay $\Delta\tau_A$ to contend with. Currently, the MPG models only the delay in the Earth atmosphere. It is possible, however, that future versions of the MPG could implement delays due to other planetary atmospheres, should requirements so demand. MPG atmospheric delay prediction is discussed later in this chapter.

As for delays in the transmitter $\Delta\tau_{D,T}$ and receiver $\Delta\tau_{D,R}$ electronics, these are tabulated for spacecraft and Earth stations, and can be applied to the overall delay. These parameters are not used in the MPG, but are applied in JPL's orbit determination programs.

Finally, there are the effects of charged particles in the ray path. These have the effect of advancing the phase of the carrier while delaying the light time. Such effects are accounted in JPL's Orbit Determination Program (ODP), but are not included in MPG predictions.

4.3.3 The Light Time Equation and its Solution

The determination of light time boils down to determining the ephemeris times of transmission and reception. In the absence of general relativistic effects, the Newtonian light time in the SSB frame would be found by dividing the magnitude of the difference in vector position (i.e., distance) by the speed of light,

$$LT_N(t_T,t_R)=\frac{|\mathbf{p}_R(t_R)-\mathbf{p}_T(t_T)|}{c}=\frac{|\mathbf{p}_R^T|}{c} \quad (4\text{-}7)$$

[4] The calculation of state vectors may, in general, require reading several ephemerides. Fortunately, SPICE routines handle such extractions invisibly to the MPG.

One immediately notes that the light time depends on positions at points in time to be solved for. One of these times may be chosen as the instant of interest, and the other then becomes a function of light time, as $t_R = t_T + LT$. The equation above then becomes a form with the light time as the independent variable,

$$LT = \frac{|\mathbf{p}_R(t_T + LT) - \mathbf{p}_T(t_T)|}{c} \tag{4-8}$$

This is the so-called forward light time equation. The backward light time equation results when t_R is chosen as the instant of interest.

$$LT = \frac{|\mathbf{p}_R(t_R) - \mathbf{p}_T(t_R - LT)|}{c} \tag{4-9}$$

Adding in gravitational delays produces the following light time equation[5] in the SSB frame

$$LT = t_R - t_T = LT_N(t_T, t_R) + \Delta LT(t_T, t_R) \tag{4-10}$$

This light time formulation does not contain the effects of atmospheric and other delay corrections, which will be addressed subsequently. The forward light time equation in the SSB frame is then

$$LT = \frac{|\mathbf{p}_T(t_T) - \mathbf{p}_R(t_T + LT)|}{c} + \Delta LT(t_T, t_T + LT) \tag{4-11}$$

which is summed over the B bodies of influence. The backwards light time equation results under the substitution $t_T = t_R - LT$. The nominal set of sources consists of the Sun, Earth, Moon, and barycenters of the other planetary systems in the solar system. The solution of this equation is fairly straightforward. Starting with a zero estimate of LT, the right hand side of the equation is evaluated to produce the next estimate. This procedure continues until the difference between estimates falls below the DSN accuracy goal.

Fortunately for the MPG, the NAIF utility `RLTIME` solves both of the SSB frame light time equations, given the identifiers of the transmitting and receiving entities, the ephemeris time at one of these entities, and the direction in which the equation is to be solved. Bodies of influence are prespecified in a kernel pool

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[5] This equation is derived in Moyer2000, Section 8.

within the SPICE system. The interested reader may consult the commentary in the code.

It remains only to compute the MPG light time, which includes the effects atmospheric and other delays. Strictly speaking, these should have been included in the computation of indicated in Eq.(4-11), rather than treating them in later. But `RLTIME` solves Eq.(4-11), so accommodations have to be made.

Let it be supposed that the sum of unaccounted delays is δ and that LT_0 represents the light time solution to Eq.(4-11) with δ excluded. That is, LT_0 is the `RLTIME` solution. Now the true light time will be $LT_0 + \delta t$, where δt is the offset due to δ. The forward light time equation becomes

$$LT_0 + \delta t = LT_N(t_T, t_T + LT_0 + \delta t) + \Delta LT(t_T, t_T + LT_0 + \delta t) + \delta \quad (4\text{-}12)$$

A first-order Taylor series expansion in δt produces the approximation

$$\begin{aligned} LT_0 + \delta t &= LT_N(t_T, t_T + LT_0) + \Delta LT(t_T, t_T + LT_0) + \delta \\ &\quad + \delta t(L\dot{T}_N(t_T, t_T + LT_0) + \Delta L\dot{T}(t_T, t_T + LT_0)) + \cdots \end{aligned} \quad (4\text{-}13)$$

Upon removing the LT_0 solution and collecting terms, the solution for δt becomes

$$\begin{aligned} \delta t &= \frac{\delta}{1 - L\dot{T}_N(t_T, t_T + LT_0) - \Delta L\dot{T}(t_T, t_T + LT_0)} \\ &= \frac{\delta}{1 - L\dot{T}_0} \approx \delta \end{aligned} \quad (4\text{-}14)$$

The light time derivative is approximately equal to the derivative of the Newtonian term, whose magnitude is roughly the separation velocity divided by the speed of light. As an extreme example, for a transmitter flying by Mercury and receiver on Earth, the error in omitting the light time derivative is less than 3×10^{-4}. This is far more accurate than any of the models of atmospheric delays available today. Moreover, an extreme atmospheric delay equivalent to 1km would only incur about 0.1ns error under the approximation.

Finally, then, the end-to-end ephemeris and atomic light times are well approximated by

$$\begin{aligned} LT &= LT_0 + \Delta\tau_A(\varepsilon) \\ L\tau &= (\tau_R - t_R) + LT_0 + (t_T - \tau_T) + \Delta\tau_A(\varepsilon) \end{aligned} \tag{4-15}$$

The atmospheric delay notation here displays its dependence on DSS antenna elevation. The time that the elevation is evaluated depends on whether the DSS antenna is transmitting or receiving.

4.3.4 Light Time Derivative Equation and its Solution

Computation of the ephemeris light time derivative is dependent on which end of the transmission is designated as the observer. The two alternatives are

$$\begin{aligned} \frac{d\,LT}{dt_T} &= \frac{dt_R}{dt_T} - 1 \\ \frac{d\,LT}{dt_R} &= 1 - \frac{dt_T}{dt_R} \end{aligned} \tag{4-16}$$

Elimination of the transmitter-receiver time derivative produces the pair of results

$$\begin{aligned} L\dot{T}_R &= \frac{L\dot{T}_T}{1 + L\dot{T}_T} \\ L\dot{T}_T &= \frac{L\dot{T}_R}{1 - L\dot{T}_R} \end{aligned} \tag{4-17}$$

where $L\dot{T}_i = dLT\,/\,dt_i$. The two observed derivatives are thus directly related to each other. When appearing without the subscript, the light time derivative will be assumed to that observed at the receiver.

The equations above also apply to atomic light times, as

$$\begin{aligned} L\dot{\tau}_R &= \frac{L\dot{\tau}_T}{1 + L\dot{\tau}_T} \\ L\dot{\tau}_T &= \frac{L\dot{\tau}_R}{1 - L\dot{\tau}_R} \end{aligned} \tag{4-18}$$

Evaluation of the ephemeris light time derivative follows straightforwardly from Eq.(4-11). In particular, because of the appearance of light time on both sides of the equation, differentiation will yield an expression of the form

$$L\dot{T}_i = C_0 + C_i \, L\dot{T}_i \tag{4-19}$$

From this is determined the derivative value,

$$L\dot{T}_i = \frac{C_0}{1 - C_i} \tag{4-20}$$

For example, considering only the Newtonian term of Eq.(4-11) produces

$$\begin{aligned} L\dot{T}_T &= \frac{1}{c|\,\mathbf{p}_R^T\,|}\Big[(\mathbf{p}_R^T \bullet \mathbf{v}_R^T) + (\mathbf{p}_R^T \bullet \mathbf{v}_R^B)\, L\dot{T}_T\,\Big] \\ L\dot{T}_R &= \frac{1}{c|\,\mathbf{p}_R^T\,|}\Big[(\mathbf{p}_R^T \bullet \mathbf{v}_R^T) - (\mathbf{p}_R^T \bullet \mathbf{v}_T^B)\, L\dot{T}_R\,\Big] \end{aligned} \tag{4-21}$$

The special utility `RLTIME` computes the gravity-corrected versions of light time derivative.

4.4 Light Time Predictions

This section discusses the light time data types calculated by the MPG. It also discusses models of atmospheric delays, so the entire method of light time calculation is contained in this chapter.

4.4.1 One-Way Light Time

One-way light times apply to simple transmissions from a DSS to a spacecraft and from the spacecraft to a DSS. The two DSSs need not be the same, and the transmission by the spacecraft need not be related to a previous uplink reception. One-way light times are used, among other things, for predicting the proper uplink transmitted frequency to a spacecraft so as to arrive at its receiver's known center frequency, for predicting the received frequency from a spacecraft transmitting in open-loop mode, for arraying several DSS antennas toward the same target, and for very long baseline interferometry (VLBI).

The one-way uplink (command) light time is defined as

$$\begin{aligned} L\tau_{12} &= \tau_2 - \tau_1 \\ &= (\tau_2 - t_2) + (t_2 - t_1) + (t_1 - \tau_1) \\ &= LT_{12} + \Delta t_2 + \Delta\tau_1 + \Delta\tau_{A,1}(\varepsilon) \end{aligned} \tag{4-22}$$

where τ_1 is TAI at an instant of transmission at the DSS, and τ_2 is TAS at the corresponding instant of reception at the spacecraft.

The one-way downlink ("observed") light time is similarly defined as

$$\begin{aligned} L\tau_{23} &= \tau_3 - \tau_2 \\ &= (\tau_3 - t_3) + (t_3 - t_2) + (t_2 - \tau_2) \\ &= LT_{23} - \Delta\tau_3 - \Delta t_2 + \Delta\tau_{A,3}(\varepsilon) \end{aligned} \tag{4-23}$$

where τ_2 is TAS at is the instant of transmission at the spacecraft, and τ_3 is TAI at the corresponding instant of reception at a DSS. As noted earlier, TAS predictions are accurate within an unknown constant offset. One-way light time prediction applications, however, are generally not sensitive to constant offsets.

4.4.2 Two-Way Light Time

A two-way configuration is one in which a single DSS both transmits and receives from a spacecraft. Insofar as two-way light time is concerned, the spacecraft downlink need not be coherent with the uplink. In the two-way geometry, the locations of the clocks at transmission and reception are the same. The observed round-trip light time (RTLT) is the difference

$$L\tau_{13} = \tau_3 - \tau_1 \tag{4-24}$$

The same nomenclature of variables as used in the previous subsection is also used here. In this case, the two-way light time is truly an observed phenomenon at the DSS.

In a two-way situation, neither the instant of the photon's arrival at the spacecraft and simultaneous departure of another back toward the DSS, nor the spacecraft timescale, enter into the computation. However, in practicality, RTLT is computed as the sum of two one-way light times,

$$\begin{aligned} L\tau_{13} &= L\tau_{12} + L\tau_{23} \\ &= (\tau_3 - \tau_2) + (\tau_2 - \tau_1) \\ &= LT_{12} + LT_{23} + (\tau_3 - t_3) - (\tau_1 - t_1) + \Delta\tau_{A,1}(\varepsilon) + \Delta\tau_{A,3}(\varepsilon) \end{aligned} \tag{4-25}$$

It is not necessary to use TAS at the spacecraft in this computation. The ephemeris light times combine with the difference in DSS-SSB time translation over the transit time.

Observed RTLT is referenced to τ_3, whereas command RTLT is referenced to τ_1. Computation of the component ephemeris light times using `RLTIME` requires the forward direction in command RTLT and the backward direction for observed RTLT. Atomic-to-ephemeris and ephemeris-to-atomic time conversions are applied according to the direction selected.

4.4.3 Three-Way Light Time

A three-way configuration is one in which one DSS transmits to a spacecraft, but another DSS receives the signal returned from the spacecraft. Insofar as 3-way light time prediction is concerned, the spacecraft downlink need not be coherent with the uplink.

In a MPG 3-way light time prediction, the time of a photon's departure is reckoned at the transmitting station's clock, but the subsequent time of arrival of the companion photon from the spacecraft is reckoned at the receiving station's clock. By the convention established earlier, the transmitter DSS frame conversion of t_1 from ephemeris time to atomic time of that frame will yield $\tau_{1,T}$. Similarly, the receiver DSS frame conversion of t_3 from ephemeris time to atomic time of that frame will yield $\tau_{3,R}$. The T and R subscripts are maintained here to emphasize that they represent different frames of reference. The MPG 3-way light time is then

$$
\begin{aligned}
L\tau_{1(T)3(R)} &= \tau_{3,R} - \tau_{1,T} \\
&= LT_{1(T)2} + LT_{23(R)} + (\tau_{3,R} - t_3) - (\tau_{1,T} - t_1) \\
&\quad + \Delta\tau_{A,1(T)}(\varepsilon) + \Delta\tau_{A,3(R)}(\varepsilon)
\end{aligned}
\qquad (4\text{-}26)
$$

Observable and command three-way light time solutions observe the same forward and backward directivities as do two-way light times.

Strictly speaking, light time calculated in this manner is not an observable quantity, because observances in two frames appear in the measure.

The reason that the MPG computes this form of 3-way light time metric is that it predicts when events initiated at the transmitting DSS become effective at the receiving DSS. Questions of simultaneity are not at issue, as the two stations are not comparing their own observations of light time.

If they were to do so, for example, the 3-way light time as observed in the transmitting station's frame would be

$$L\tau_{13,T} = \tau_{3,T} - \tau_{1,T} \qquad (4\text{-}27)$$

and in the receiving DSS frame, the observed 3-way light time would be

$$L\tau_{13,R} = \tau_{3,R} - \tau_{1,R} \qquad (4\text{-}28)$$

These two measurements, both consistent with the laws of relativity, are not the same, but different by an amount

$$\Delta L\tau_{13} = (\tau_{3,T} - \tau_{3,R}) - (\tau_{1,T} - \tau_{1,R}) \qquad (4\text{-}29)$$

This difference between the two station's observances of light time bears out the rule of relativity that phenomena recorded "simultaneously" in two frames may appear to be different.

4.4.4 Atmospheric Delay Computation

Modeling the delay of an electronic wave as it traverses the neutral[6] atmosphere is termed *tropospheric delay* and is generally divided into two components, which have been designated the wet and hydrostatic (dry) components along the line-of-sight path. The wet delay is caused by the permanent dipole in atmospheric water vapor, and hydrostatic or dry delay is, caused by induced dipoles in all atmospheric gasses. The zenith dry delay ranges from about 2 to 2.2 meters at DSS sites. DSS zenith wet delays are smaller and more variable, ranging from a few centimeters to 25 centimeters or more.

Tropospheric delay is a function of antenna elevation because the wave traverses a longer path through the atmosphere when arriving or departing at lower elevations than it does when arriving or departing from the zenith. The slant-range factor is approximately 10 at 6 degrees elevation. Since tropospheric delay is practically symmetric with respect to azimuth, tropospheric delay predictions are not specific to any particular spacecraft.

Wet and dry delays are modeled separately, each as the product of a zenith delay and a geometric function of antenna elevation angle, called the *mapping function*, here denoted $m(\varepsilon)$, where ε is the elevation angle. The predicted atmospheric delay as a function of antenna elevation ε takes the form

$$\Delta\tau_A(\varepsilon) = \Delta\tau_{A,H}(90°)\, m_H(\varepsilon) + \Delta\tau_{A,W}(90°)\, m_W(\varepsilon) \qquad (4\text{-}30)$$

The MPG receives zenith delay estimates in the form of seasonal models at each Deep Space Complex that do not depend on real time data and need only to be redelivered when climate changes become significant. These models represent the one-way zenith wet and dry delays, in meters, as Fourier series expansions of the corrections.

Data received from the topocentric calibration file consist of start (S) and end (E) times that the series is applicable, the designated deep space complex, and the Fourier period P and coefficients $(a_0, a_1, b_1, a_2, b_2, \cdots)$. The period is typically one year. Each zenith delay at the time T is then calculated as

[6] Delay due to charged particles in the atmosphere is not modeled within the MPG.

$$\begin{aligned} \Delta\tau_A(90^\circ) &= a_0 + a_1\sin(X) + b_1\cos(X) + a_2\sin(2X) + b_2\cos(2X) + \cdots \\ X &= \frac{2\pi(T-S)}{P} \end{aligned} \tag{4-31}$$

The zenith dry delay is known to decrease by about 0.25 mm for each meter of altitude from the deep space complex. Dry delay calibrations refer to the altitude of the antenna datum at each DSS. Constant offsets for each DSS are read from the tropospheric calibration file and applied to the zenith delay. Further information regarding tropospheric calibration file may be found in Runge2000.

The mapping function used by the MPG is one developed by C. C. Chao in 1973 (see Chao1973). It takes the form

$$m(\varepsilon) = \frac{1}{\sin(\varepsilon) + \dfrac{A}{\tan(\varepsilon) + B}} = \frac{\sin(\varepsilon) + B\cos(\varepsilon)}{\sin^2(\varepsilon) + \cos(\varepsilon)(A + B\sin(\varepsilon))} \tag{4-32}$$

The second form of the function is preferred for computations, since it does not involve the infinite zenith tangent. The value is unity at zenith. The derivative, with the assistance of *Mathematica*, is straightforwardly found to be

$$m'(\varepsilon) = m^2(\varepsilon)\left(\frac{A}{(\sin(\varepsilon) + B\cos(\varepsilon))^2} - \cos(\varepsilon)\right) \tag{4-33}$$

Different values for A and B apply to wet and dry delays,

$$\begin{aligned} A_H &= 0.00147 \qquad & A_W &= 0.00035 \\ B_H &= 0.04000 \qquad & A_W &= 0.01700 \end{aligned} \tag{4-34}$$

4.5 The Light Time-Frequency Relationship

Deep space transmissions take the form of sinusoidal waves modulated with command, telemetry, and ranging codes. The received signal can, by design, be separated into its distinct parts, consisting of carrier and various forms of modulation. The definition of frequency used in the DSN is the cyclic rate of change of the carrier phase. Thus, if a signal bears a carrier component $A\sin(\theta(\tau))$ then the frequency of the signal is

$$f(\tau) = \frac{1}{2\pi}\frac{d\theta(\tau)}{d\tau} \tag{4-35}$$

where the derivative is taken with respect to the atomic clock at the point of interest, i.e., at the transmitter or receiver. The phase function is often envisioned as a linear angular change, such as $2\pi f + \phi$, where f is the frequency and ϕ is the phase offset. But in the more general sense, frequency is the derivative of the carrier phase function divided by 2π.

If $\theta_T(\tau_T)$ represents the transmitted phase function as a function of its atomic timescale, then the received phase function will be of the form

$$\theta_R(\tau_R) = \theta_T(\tau_T + L\tau) + \phi(\tau_R) \tag{4-36}$$

in which $L\tau$ is the apparent time delay between transmitted and received atomic clocks and $\phi(\tau_T)$ is any apparent rotation of the carrier phase not attributable to the light time delay, such as might be due to the effects of charged particles along the flight path, or due to relative rotation between the axes of the two antennas during the time of flight.

Whereas the ODP does contain models for estimating that part of $\phi(\tau_R)$ due to relative antenna motion using information[7] on spacecraft orientation, these are not currently used in MPG predictions. Thus, the MPG presumes that the transmitted and received phase functions are the same, but disagree only in time argument and timescale. MPG frequency predictions are based on the formulas

$$\begin{aligned} f_T(\tau_T) &= \frac{1}{2\pi}\frac{d\theta(\tau_T)}{d\tau_T} \\ f_R(\tau_R) &= \frac{1}{2\pi}\frac{d\theta(\tau_R)}{d\tau_R} \\ \tau_R &= \tau_T + L\tau \end{aligned} \tag{4-37}$$

[7] This information is available in SPICE C-kernel files.

4.5.1 Frequency Relationships

The ratio of received and transmitted frequencies is then found using the chain rule,

$$\frac{f_R}{f_T} = \frac{\dfrac{d\theta}{d\tau_R}}{\dfrac{d\theta}{d\tau_T}} = \frac{d\tau_T}{d\tau_R} = 1 - \frac{dL\tau}{d\tau_R} = 1 - L\dot{\tau} \tag{4-38}$$

where $L\dot{\tau}$, by convention, refers to the derivative with respect to proper time at the receiver, i.e, $L\dot{\tau}_R$.

The relationships between transmitted and received frequencies may be then written so as to separate the effects of the transmission medium and timescale translation,

$$\begin{aligned} f_R &= f_T - Y\, f_T = (1 - Y) f_T \\ f_T &= f_R + \overline{Y}\, f_R = (1 + \overline{Y}) f_R \\ Y &= L\dot{\tau} = L\dot{\tau}_R \\ \overline{Y} &= \frac{L\dot{\tau}_R}{1 - L\dot{\tau}_R} = L\dot{\tau}_T \end{aligned} \tag{4-39}$$

The coefficient Y is essentially the Doppler factor, so its magnitude is approximately the relative velocity between transmitter and receiver scaled by the speed of light. The notations Y and $\overline{Y}$ were introduced in a working document communicated to the authors by Nicole Rappaport [Rappaport2001]; however, the formulation in Eq. (4-39) is expressed directly in the light time derivative, rather than in the Rappaport parameters. Moreover, it is apparent from Eq.(4-18) that $\bar{Y} = L\dot{\tau}_T$.

The forms above permit the Doppler estimates to be generated separately, and then combined with the base frequency, thus achieving better numeric accuracy. The MPG has a frequency accuracy specification of 500 μHz at Ka-band (32 gHz), which is a relative accuracy of 1.5625×10^{-14} . This is dangerously close to the limits of double precision numeric accuracy on MPG host computers, so care is required in order not to violate the specification.

4.5.2 Light Time Derivative

Computation of Doppler profiles requires evaluation of the light time derivative. The derivative of $L\tau = \tau_R - \tau_T$ with respect to the received time is clearly

$$L\dot{\tau}_R = 1 - \frac{d\tau_T}{d\tau_R} = 1 - \frac{\dfrac{d\tau_T}{dt_T}}{\dfrac{d\tau_R}{dt_R}} \frac{dt_T}{dt_R} \tag{4-40}$$

It is immediately recognized that

$$\begin{aligned} \frac{dt_T}{dt_R} &= 1 - L\dot{T}_R \\ \frac{d\tau_T}{dt_T} &= 1 + D_{t_T} \\ \frac{d\tau_R}{dt_R} &= 1 + D_{t_R} \end{aligned} \tag{4-41}$$

Note that the ephemeris light time derivative $L\dot{T}_R$ is with respect to the ephemeris time at the receiver, whereas $L\dot{\tau}_R$ is the proper light time with respect to atomic time at the receiver. Substitution produces

$$L\dot{\tau}_R = L\dot{T}_R + \left(\frac{D_{t_R} - D_{t_T}}{1 + D_{t_R}} \right) (1 - L\dot{T}_R) \tag{4-42}$$

The ephemeris light time derivative is found from Eq.(4-14), which yields the result

$$\begin{aligned} L\dot{T}_R &= L\dot{T}_{R,0} + \Delta\tau'_A \\ \Delta\tau'_A &= \Delta\tau'_A(\varepsilon)\ \dot{\varepsilon} \end{aligned} \tag{4-43}$$

This calculation requires the derivative of the ephemeris light time (returned by `RLTIME`), the derivative of the atmospheric model with respect to elevation as in Eq.(4-33), as well as the time rate of change of the elevation itself (treated elsewhere in this Supplement).

4.6 Frequency Predictions

Frequencies allocated to deep space missions are grouped into three narrow ranges called S-band, X-band, and Ka-band. S-band is in the neighborhood of about 2200 MHz, X-band is about 8000 MHz, and Ka-band is about 32 GHz. In order to avoid passband interference, transmitted and received frequencies in each band are separated by an amount found to be sufficient for that task.

Some spacecraft have the capability to maintain phase coherence between uplink and downlink carriers. The onboard transponder generates the downlink carrier wave by subjecting the incoming carrier wave to a series of frequency divisions and multiplications, all by integer values. This *transponder turnaround ratio* is discussed later in this section.

When the spacecraft transmitter frequency is not generated using the uplink carrier, but derived from a free-running oscillator, the transmitter is said to be operating in a non-coherent mode. The frequency may be one generated as above, but with no uplink, or it may come from an auxiliary oscillator, or, perhaps an ultra-stable oscillator found in some spacecraft systems.

Some spacecraft do not operate in coherent mode; their transmissions are always non-coherent, whether the receiver is in lock or out of lock. To avoid the operational problem of discerning, at a ground station, whether or not the uplink is being received by the spacecraft, a different frequency is transmitted when the onboard receiver is locked than when it is not. In this way, station personnel can tell in one light time whether or not the uplink has been acquired.

The transmitter frequency at a tracking station on Earth can be constant or ramped. Ramping refers to contiguous segments of linearly increasing frequency transmissions. If ramped, frequency values are specified in the form of a set of linear polynomials formatted in the usual Everett form, with second- and fourth-order terms equal to zero. Generation of the frequency ramp profile is treated later in this chapter.

The project's PP file contains a complete definition of frequency bands, their names, the spacecraft Best Lock Frequency (BFA), the transmitter frequency (TFREQ), the Two-Way Non-Coherent TFREQ, and all turnaround ratios that the spacecraft supports.

4.6.1 Doppler-Compensating Uplink Frequency

In order to assist carrier acquisition at the spacecraft, the MPG generates a polynomial profile of that frequency which, when distorted by Doppler, reaches the spacecraft precisely at the center of its lock-in range, termed the Best Lock Frequency (BLF), here denoted f_{BL}. This Doppler-compensated profile is referred to as XA, a historical, forgotten[8] acronym.

Since the uplink frequency is often ramped, as mentioned above, the XA prediction is sometimes, redundantly, referred to as the "unramped XA." The profile of ramped frequency that follows the XA within specified deviation criteria is referred to as the "ramped XA".

The XA computation follows the previous formulation given in Eq.(4-39), as detailed below.

$$\begin{aligned} f_{XA} &= f_{BL} + \bar{Y}_{12}\, f_{BL} \\ Y_{12} &= L\dot{\tau}_{12(2)} \\ \bar{Y}_{12} &= \frac{Y_{12(2)}}{1 - Y_{12(2)}} \\ L\dot{\tau}_{12(2)} &= L\dot{T}_{12(2)} + \left(\frac{D_{t_2} - D_{t_1}}{1 + D_{t_2}}\right)(1 - L\dot{T}_{12(2)}) \\ &\approx L\dot{T}_{12(2)} + (D_{t_2} - D_{t_1})(1 - L\dot{T}_{12(2)}) \end{aligned} \tag{4-44}$$

The (2) subscripts above indicate that the observer is at the receiver, and that the ephemeris light time derivative appearing in the expression is that obtained from `RLTIME` under this calling convention. The approximation appearing here is correct within terms of order $O(c^{-4})$, which is the same accuracy with which atomic/ephemeris time conversions are made.

The reader should note at this point that the frequency f_{XA} to be transmitted is, in the form above, computed with respect to, and therefore tabulated against, light time derivatives at the receiver, not at the transmitter. Fortunately, Eq.(4-18)

[8]At one time this prediction was generated for the digitally controlled oscillator that then fed the uplink exciter, which was then multiplied up to sky frequency for transmission. It is the author's recollection of the early DSN that the X in XA derived from this usage. The A is believed to refer to the exciter being "adjusted" to the Doppler profile. In the MPG XA is output at sky frequency.

provides the means for converting Eq.(4-44) into terms observed at the transmitter, which is

$$
\begin{aligned}
f_{XA} &= f_{BL} + \overline{Y}_{12}\, f_{BL} \\
\overline{Y}_{12} &= \frac{Y_{12}}{1 - Y_{12}} = L\dot{\tau}_{12(1)} \\
L\dot{\tau}_{12(1)} &= L\dot{T}_{12(1)} + \left(\frac{D_{t_2} - D_{t_1}}{1 + D_{t_1}} \right)(1 + L\dot{T}_{12(1)}) \\
&\approx L\dot{T}_{12(1)} + (D_{t_2} - D_{t_1})(1 + L\dot{T}_{12(1)})
\end{aligned}
\tag{4-45}
$$

The (1) subscripts above indicate that the observer is at the transmitter, and that the ephemeris light time derivative appearing in the expression is that obtained from `RLTIME` under this calling convention. The approximation appearing here is again correct within terms of order $O(c^{-4})$.

4.6.2 One-Way Non-Coherent Downlink Frequency

The spacecraft transmitter frequency known as TFREQ is the one-way non-coherent frequency of the onboard oscillator. As mentioned above, there may be a number of such frequencies depending on the set of on-board oscillators. Here, the non-coherent downlink transmitted frequency will merely be denoted as $f_{2,T}$. The frequency received at the Earth station will then follow Eq.(4-39), as detailed below.

$$
\begin{aligned}
f_3 &= f_{2,T} - Y_{23}\, f_{2,T} \\
Y_{23} &= L\dot{\tau}_{23} \\
L\dot{\tau}_{23} &= L\dot{T}_{23} + \left(\frac{D_{t_3} - D_{t_2}}{1 + D_{t_3}} \right)(1 - L\dot{T}_{23}) \\
L\dot{T}_{23} &= L\dot{T}_{0,23} + \Delta\tau'_{A,3}
\end{aligned}
\tag{4-46}
$$

4.6.3 Two-Way and Three-Way Non-Coherent Frequencies

Spacecraft that do not communicate coherently indicate that they have locked onto the uplink by switching from one on board oscillator, running at TFREQ, to another, running at TFREQ_TWNC (transmitted frequency, two-way, non-coherent). Likewise, they revert to TFREQ when the uplink receiver is not in lock. The net effect is to make this type of spacecraft respond at ground stations like coherent ones, in that the downlink received frequency shifts as the onboard receiver goes into and out of lock.

However, there is no two-way or three-way Doppler imposed on the received downlink carrier. Only the one-way downlink Doppler appears, whose mathematical behavior is the same as that appearing in Eq. (4-46).

4.6.4 Two-Way and Three-Way Coherent Frequencies

In two-way and three-way coherent links, the uplink received frequency is, by Eq.(4-39),

$$\begin{aligned} f_{2,R} &= f_1 - Y_{12}\, f_1 \\ Y_{12} &= L\dot{\tau}_{12} \end{aligned} \tag{4-47}$$

Coherent downlink transmission occurs when the spacecraft translates its received carrier phase to another that has been shifted well out of the spacecraft receiver's passband by multiplying it by a constant factor R, the transponder turnaround ratio, whose values are given in later in this chapter.

The spacecraft transmits the frequency $f_{2,T} = R\, f_{2,R}$, so that the frequency received back at the ground station follows Eq.(4-46) evaluated using this value,

$$\begin{aligned} f_3 &= f_{2,T} - Y_{23}\, f_{2,T} = (1 - Y_{23})\, f_{2,T} = R\,(1 - Y_{23})(1 - Y_{12})\, f_1 \\ &= R\, f_1 - Y_{13}\, R\, f_1 = (1 - Y_{13}) R\, f_1 \end{aligned} \tag{4-48}$$

Here, Y_{13} represents the combined two-way or three-way coherent Doppler factor,

$$Y_{13} = 1 - (1 - Y_{12})(1 - Y_{23}) = Y_{12} + Y_{23} - Y_{12} Y_{23} \tag{4-49}$$

The Y_{13} factor permits the smaller effects imposed by the transmission medium to be separated from the larger frequency parameters, thus potentially increasing accuracy of calculations. Uplink and downlink effects are calculated separately and combined to yield the desired result.

4.6.5 Two-Way and Three-Way Doppler Compensation

It is possible to extend the concept of spacecraft Doppler compensation as appears in section 4.6.1 to apply to two-way and three-way coherent configurations, as well. The frequency relationships between received and transmitted frequencies given in Eq.(4-48) can be inverted to relate the transmitted frequency, again denoted as f_{XA}, to the desired received frequency, here again denoted f_{BL},

$$\begin{aligned} f_{XA,1} &= \frac{1}{(1-Y_{13})R} f_{BL,3} = (1+\bar{Y}_{13})\frac{f_{BL,3}}{R} \\ \bar{Y}_{13} &= \frac{Y_{13}}{1-Y_{13}} \end{aligned} \tag{4-50}$$

The Y_{13} in the above equation is that found in Eq.(4-49), which is, of course, expressed in receiver-observer values. When expressed in transmitter-observer light-derivative terms, the 2-way and 3-way XA frequency becomes

$$\bar{Y}_{13} = L\dot{\tau}_{12(1)} + L\dot{\tau}_{23(2)} + L\dot{\tau}_{12(1)} L\dot{\tau}_{23(2)} \tag{4-51}$$

which is, of course, merely the transmitter-receiver observer translation of Y_{13}. The added subscripts indicate the index of the observer in each leg.

4.6.6 Transponder Turnaround Ratios

Spacecraft transmit on frequencies that are outside the receiver's passband to avoid interference. When the uplink and downlink carrier frequencies belong to different bands, this presents no problem. But when they belong to the same band, the separation of frequencies within the band must be carefully designed.

The onboard transponder electronics generates a downlink frequency that is phase coherent with the received uplink carrier by use of frequency multiplication and division by integer values, usually in several stages. The particular ratio is a function of the uplink and downlink carrier bands chosen for operation, and the numerator and denominator integers are chosen not only to separate the two frequencies, but also to simplify the design of the transponder.

The table below enumerates the standard DSN turnaround ratios for S-, X-, and Ka-band uplink/downlink combinations. The actual turnaround ratios that the spacecraft supports is found in the project's PP file.

Table 1. DSN Standard Turnaround Ratios

Uplink Frequency	Downlink Frequency		
	S	X	Ka
S	$\frac{240}{221}$	$\frac{880}{221}$	$\frac{3344}{221}$
X	$\frac{240}{749}$	$\frac{880}{749}$	$\frac{3344}{749}$
Ka	$\frac{240}{3599}$	$\frac{880}{3599}$	$\frac{3344}{3599}$

4.6.7 Generation of Ramped Frequency

As mentioned earlier, the transmitter frequency at a tracking station on Earth can be broken into contiguous segments of linearly increasing frequency transmissions, called frequency ramps.

Ideally, the DSN might prefer to transmit f_{XA} directly to the spacecraft so that its receiver would always receive its input at the optimum point for phase lock acquisition and least loop stress thereafter. However, current DSS frequency synthesizers are incapable of being programmed to follow f_{XA} with high precision. Rather, they are capable of generating a varying frequency profile in a phase coherent manner only in the case where the frequency changes linearly over time.

The MPG is thus tasked to produce frequency predictions that are linear approximations to the f_{XA} characteristic within prespecified deviation criteria. Two types of ramping are allowed. The first is termed *ramp-on-curve*, and refers to a profile of contiguous linear segments that begin and end exactly on the f_{XA}

curve. The other is termed *ramp-off-curve*, and refers to a profile of contiguous linear segments that begin and nominally end within an allowable deviation from f_{XA}.

To be useful, each series of linear frequency ramps is required to follow the f_{XA} characteristic within a given maximum deviation, here denoted $\Delta f_{\max}$. This seemingly simple task is actually fairly complex. The remainder of this section addresses the means used for satisfying the requirement.

4.6.7.1 Mathematical Basis

Let $(\tau_0, \tau_1, \tau_2, \cdots \tau_N)$ denote the list of instants of proper time at the transmitter that define the beginning and ending times of each ramp. The subscript "1" denoting that these are times on the transmitter clock have been omitted for brevity. It will be assumed that $f_{XA}(\tau)$ is continuous and has continuous derivatives in each interval (τ_i, τ_{i+1}), or can be divided into non-overlapping intervals in which this is the case. In the latter case, each of the subintervals of continuity is treated separately.

It will be found convenient to treat intervals separately, and to translate the focus of attention on the τ axis to the beginning of the interval currently being examined. Let

$$\begin{aligned} x &= \tau - \tau_i \\ f(x) &= f_{XA}(\tau) \\ h &= \tau_{i+1} - \tau_i \end{aligned} \tag{4-52}$$

The interval defined by (τ_i, τ_{i+1}) corresponds to a linear change in frequency given by the ramp

$$ramp(x) = f(0) - \Delta_0 + m\,x \tag{4-53}$$

Interval length h, slope m, and initial offset Δ_0 are required to satisfy the condition

$$|\; ramp(x) - f(x) \;| \leq \Delta f_{\max} \tag{4-54}$$

for all x within the interval $(0,h)$, with h to be determined. The value of Δ_0 is an input parameter, whose value is zero for the ramp-on-curve case, and $\pm\Delta f_{\max}$ in the ramp-off-curve case.

4.6.7.2 The Deviation Equation

The difference between a candidate ramp and f_{XA} over the interval of interest is

$$\delta(x) = f(x) - ramp(x) = f(x) - (f(0) - \Delta_0 + m\,x) \qquad (4\text{-}55)$$

The end offset Δ_h is the deviation at the end of the interval,

$$\Delta_h = \delta(h) = f(h) - f(0) + \Delta_0 - m\,h \qquad (4\text{-}56)$$

For the ramp-on-curve case, $\Delta_0 = \Delta_h = 0$. For the ramp-off-curve case, the magnitudes of the endpoint offsets would ideally both be exactly equal to $\Delta f_{\max}$. However, in any case, the slope m can be solved for from the above and found to be

$$m = \frac{(f(h) - f(0)) - (\Delta_h - \Delta_0)}{h} \qquad (4\text{-}57)$$

It remains only to determine h when given $f(0)$ and the initial offset Δ_0 for the ramp-on/off-curve case being profiled. Having found the interval length, the initial conditions for the next interval will then have already been determined, since finding h entails determining $f(h)$ and Δ_h. Thus, each next interval length may be determined in the same manner, until the entire time span has been ramp-fitted.

The sign of the initial offset of the first segment of the ramp-off-curve case may be chosen freely. The initial and final offsets of the remainder of the segments are determined in sequence.

The solution for the interval length is subject to the stipulation

$$|\ \delta(x)\ | \le \Delta f_{\max} \quad \text{for } x \in (0,h) \qquad (4\text{-}58)$$

4.6.7.3 Solving for the Interval Length

It is always possible to satisfy (4-58) by making the interval length small enough, and it is always possible to expand this length until it is violated (except in degenerate cases, such as constant f_{XA}). The method for finding an appropriate interval length is thus an iterative one in which candidate lengths are sequentially considered. When the analysis finds that a length violates (4-58), a shorter one is tested next; when tests show that the length satisfies (4-58) short of equality, a longer one is tried. This process continues until a suitable value is found. This procedure finds the maximum interval length satisfying the constraint.

An earlier MPG phase developed a method that located all extrema within the interval and chose an interval length that brought all within the specified limit. The algorithm was very intricate and slow because all extrema had to be found and kept monitored as the length was converging in the iterative procedure. To its credit, however, it did maximize ramp length.

In order to reduce computational complexity, the MPG now requires that the curve between the endpoints have at most a single extremum. It has been found that this stipulation reduces ramp length, on average, by only a very modest amount. Proper choice of the interval length makes it possible to guarantee that no more than one extremum occurs. A series of tests is necessary to make this guarantee.

4.6.7.4 Initial Length Estimate

It is possible to derive an approximate expression for the length of the linear fit to the given function $f(x)$ as a starting heuristic in the ramp-fit algorithm. Suppose that the Taylor series of the function about the origin takes on the form

$$f(x) = f(0) + x\,f'(0) + \frac{x^n}{n!}f^{(n)}(0) + \cdots \tag{4-59}$$

That is, it consists of the usual first two terms, but all the other terms up to the *n*-th either vanish, or are negligible. The case in which *n* is unity is a degenerate case, as the entire function is already a ramp. It may be 2, of course, or any greater integer[9]. Then the deviation between the Taylor series ramp will be

---

[9] Derivatives up to fourth order are readily obtained by interpolating f_{XA}, which is in polynomial

$$\begin{aligned}\delta(x) &\approx \frac{x^n}{n!} f^{(n)}(0) + \Delta_0 + \frac{x}{h}(\Delta_h - \Delta_0 - \frac{h^n}{n!} f^{(n)}(0)) \\ &\approx \Delta_h + \frac{f^{(n)}(0)}{n!}(x^n - x\, h^{n-1}) + (\Delta_0 - \Delta_h)\left(1 - \frac{x}{h}\right)\end{aligned} \tag{4-60}$$

The Taylor series was used to evaluate both $f(x)$ and $f(h)$. One may readily verify that the approximation is exact at both interval endpoints. It is now straightforward to locate the internal extremum point by differentiation. The result is

$$\begin{aligned}x_{pk} &= \left(\frac{(\Delta_0 - \Delta_h)\, n! + h^n f^{(n)}(0)}{n\, h\, f^{(n)}(0)}\right)^{\frac{1}{n-1}} \\ &= \frac{h}{n^{1/(n-1)}}\left(1 + \frac{(\Delta_0 - \Delta_h)\, n!}{h^n\, f^{(n)}(0)}\right)^{\frac{1}{n-1}}\end{aligned} \tag{4-61}$$

Estimates for the interval lengths of the ramp cases will be considered separately below. As will be seen, the sign of the *n*-th derivative term is a determining factor in the ramp-off-curve case in the choice of the estimated interval length.

Equal endpoint deviation case: This case assumes $\Delta_0 = \Delta_h$, which includes the ramp-on-curve case ($\Delta_0 = 0$) and ramp-off-curve case with $\Delta_0 = \Delta_h = \pm\Delta f_{\max}$. From Eqs. (4-60) and (4-61) it follows that the location of the peak value and the peak value itself satisfy

$$\begin{aligned}x_{pk} &= \frac{h}{n^{n/(n-1)}} \\ \delta(x_{pk}) &= \Delta_0 + \frac{f^{(n)}(0)(n-1)}{n!\, n^{n/(n-1)}} h^n = \Delta_{pk} = \pm\Delta f_{\max}\end{aligned} \tag{4-62}$$

The solution for the estimated interval length in this case is

$$\hat{h} = \left(\frac{(\Delta_0 - \Delta_{pk})\, n!\, n^{n/(n-1)}}{(n-1)\, f^{(n)}(0)}\right)^{1/n} \tag{4-63}$$

In the ramp-on-curve case, when $\Delta_0 = 0$, the sign of the peak deviation will be opposite that of $f^{(n)}(0)$ and the estimated length is

form as one of the MPG prediction types. If this polynomial is linear, the ramp interval length should be made equal to the remaining interval length of the f_{XA} interval.

$$\hat{h} = \left(\frac{\Delta f_{\max}\, n!\, n^{n/(n-1)}}{(n-1)\mid f^{(n)}(0)\mid} \right)^{1/n} \tag{4-64}$$

In the ramp-off-curve case, the peak deviation must take the sign opposite to that of Δ_0 *and* the sign of Δ_0 must match that of $f^{(n)}(0)$ for there to be a solution. The freedom to choose the sign of Δ_0 only comes when solving for the first interval length. In subsequent intervals, if the signs of Δ_0 and $f^{(n)}(0)$ do not match, there can be no solution of this type. An interval with opposite endpoint deviations must be sought, or the estimate above must be considered an upper bound on the interval size.

Opposite endpoint deviation case: The ramp-off-curve case remains in which $\Delta_h = -\Delta_0 = \pm\Delta f_{\max}$ and no internal extremum is allowed, for reasons later discussed. The extremum, in this case will appear at one of the endpoints.

From Eq.(4-61), it is seen that the peak occurs at he beginning of the interval under the condition

$$\hat{h} = \left(-\frac{2\, n!\, \Delta_0}{f^{(n)}(0)} \right)^{1/n} \tag{4-65}$$

in which Δ_0 must have sign opposite that of $f^{(n)}(0)$. If this cannot be made the case, then the peak cannot occur at the beginning of the interval.

Also from Eq.(4-61), the peak occurs at the end of the interval under the condition

$$\hat{h} = \left(\frac{2\Delta_0\, n!}{(n-1)\, f^{(n)}(0)} \right)^{1/n} \tag{4-66}$$

In this case the sign of the initial offset must be the same as that of the *n*-th derivative.

Thus, only one of the length estimates applies, depending on the signs of the initial displacement and the initial *n*-th function derivative.

4.6.7.5 Endpoint Slope Conditions

Two necessary, but not sufficient, length criteria are related to the orientation of the slopes at the interval endpoints. They are easily evaluated and can be used to screen out improper lengths without more costly computation.

The first applies to the ramp-off-curve case, in which $|\Delta_0| = |\Delta_h| = \Delta f_{\max}$. The initial slope in this case must be in a direction to reduce the deviation, and the final slope must be in the direction of the final offset. For this to occur, the slope at the initial point must be of sign opposite to Δ_0, and the endpoint slope must be of the same sign as Δ_h. Alternately, the slopes at these points may be zero. The first screening condition is thus

$$\begin{aligned} \Delta_0\, \delta'(0) &\le 0 \\ \Delta_h\, \delta'(h) &\ge 0 \end{aligned} \tag{4-67}$$

If either of these conditions is violated, then the candidate interval length h is too great, and must be reduced. If both conditions are valid at the current length, then the internal structure of the deviation needs to be examined. The ramp-on-curve case, $\Delta_0 = \Delta_h = 0$, satisfies this condition by default.

Comparison of endpoint slopes is an indicator of the parity of the number of extrema. Under the condition that there is at most a single extremum in the interval (including endpoints), the parity indicator is $\delta'(0)\delta'(h)$. If this product is positive, or if only one of the derivatives is zero, there is none; when the product is negative, or if both derivatives are zero, then there is one.

Endpoint conditions involve the derivative of the deviation, which is

$$\delta'(x) = f'(x) - m = f'(x) - \frac{(f(h) - f(0)) - (\Delta_h - \Delta_0)}{h} \tag{4-68}$$

Finding the derivative $f'(x)$ presents no problem, since $f(x)$ values are found by interpolating the Everett polynomials representing $f_{XA}(\tau)$. The interpolation software is capable of returning derivative, as well as function values.

The ramp-on-curve fit case $\Delta_0 = \Delta_h = 0$ and ramp-off-curve fit case having same-sign endpoint deviations $\Delta_0 = \Delta_h \neq 0$ are the instances requiring that an extremum be sought. The solution process thus should seek an extremum deviation whenever

$$h^2\delta'(0)\,\delta'(h) = (f(0) - f(h) + h\,f'(0)) \times (f(0) - f(h) + h\,f'(h)) \le 0 \quad \text{(4-69)}$$

No extremum need be sought when (4-69) is violated. A shorter interval length is indicated.

4.6.7.6 Curvature Condition

By construction, the deviation equation takes on the required endpoint values, and endpoint slope conditions guarantee that the deviation direction is correct at the endpoints. Any violations of the deviation goal therefore occur within the interval. When the deviation slope at an endpoint is zero, the second derivative, if of the proper sign, could cause a violation in a small neighborhood of the endpoint.

In cases where an internal extremum is expected, design imposes the restriction that there be no other points of zero slope in either of the two segments divided at the midpoint peak. If such points are found, the interval is shortened until compliance results.

When no extremum is expected, design imposes the restriction that there be no points of zero slope anywhere within the interval. If such a point is located during interval validation, a successively lesser value of length will be chosen until the internal extremum disappears.

4.6.7.7 Extremum Conditions

Deviation extrema within the interval occur at points of zero derivative, which translates into the equation

$$f'(\chi) = m \quad \text{(4-70)}$$

To attain stability and robustness, the search interval may be divided into a predetermined number of sectors, and each tested for a change in sign. Then a bracketing root finder may be applied locate the root precisely.

If an extremum is expected within the interval, its value will be denoted x_{pk}. If one is not expected, but is found to occur, then the interval length is too great and must be reduced.

Having found an expected internal peak value, the deviation at this point may be calculated, and must satisfy

$$\delta(x_{pk}) \le \Delta f_{\max} \tag{4-71}$$

If this condition is not met, the interval length is too great, and must be reduced.

Having found x_{pk}, the regions $(0, x_{pk})$ and (x_{pk}, h) are searched for extrema other than at x_{pk}. If any are located, h is too large, and must be reduced.

If the condition (4-71) is met, but not in equality, the length may be increased. The search terminates when (4-71) is satisfied in equality (or within an acceptable proximity).

4.6.7.8 Implementation

The actual algorithm used in the MPG appears more complicated than the mathematical basis above might indicate due to its ability to handle both on-curve and off-curve cases and due to features to assure robustness over a general mission set. The interested reader is referred to the *Mathematica* study [Taus2003] cited in the References, below, and other *Mathematica* studies in the MPG archive that address the development and validation of the algorithm.

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