

## CHAPTER 9

### THE TWO BODY PROBLEM IN TWO DIMENSIONS

#### 9.1 *Introduction*

In this chapter we show how Kepler's laws can be derived from Newton's laws of motion and gravitation, and conservation of angular momentum, and we derive formulas for the energy and angular momentum in an orbit. We show also how to calculate the position of a planet in its orbit as a function of time. It would be foolish to embark upon this chapter without familiarity with much of the material covered in Chapter 2. The discussion here is limited to two dimensions. The corresponding problem in three dimensions, and how to calculate an ephemeris of a planet or comet in the sky, will be treated in Chapter 10.

#### 9.2 *Kepler's Laws*

Kepler's law of planetary motion (the first two announced in 1609, the third in 1619) are as follows:

1. Every planet moves around the Sun in an orbit that is an ellipse with the Sun at a focus.
2. The radius vector from Sun to planet sweeps out equal areas in equal time.
3. The squares of the periods of the planets are proportional to the cubes of their semi major axes.

The first law is a consequence of the inverse square law of gravitation. An inverse square law of attraction will actually result in a path that is a *conic section* – that is, an ellipse, a parabola or a hyperbola, although only an ellipse, of course, is a closed orbit. An inverse square law of repulsion (for example,  $\alpha$ -particles being deflected by gold nuclei in the famous Geiger-Marsden experiment) will result in a hyperbolic path. An attractive force that is directly proportional to the first power of the distance also results in an elliptical path (a Lissajous ellipse) - for example a mass whirled at the end of a Hooke's law elastic spring - but in that case the centre of attraction is at the centre of the ellipse, rather than at a focus.

We shall derive, in section 9.5, Kepler's first and third laws from an assumed inverse square law of attraction. The problem facing Newton was the opposite: Starting from Kepler's laws, what is the law of attraction governing the motions of the planets? To start with, he had to invent the differential and integral calculus. This is a far cry from the popular notion that he "discovered" gravity by seeing an apple fall from a tree.

The second law is a consequence of conservation of angular momentum, and would be valid for any law of attraction (or repulsion) as long as the force was entirely radial with no transverse component. We derive it in section 9.3.

Although a full treatment of the first and third laws awaits section 9.5, the third law is trivially easy to derive in the case of a *circular* orbit. For example, if we suppose that a planet of mass  $m$  is in a circular orbit of radius  $a$  around a Sun of mass  $M$ ,  $M$  being supposed to be so much larger than  $m$  that the Sun can be regarded as stationary, we can just equate the product of mass and centripetal acceleration of the planet,  $ma\omega^2$ , to the gravitational force between planet and Sun,  $GMm/a^2$ ; and, with the period being given by  $P = 2\pi/\omega$ , we immediately obtain the third law:

$$P^2 = \frac{4\pi^2}{GM} a^3. \quad 9.2.1$$

The reader might like to show that, if the mass of the Sun is not so high that the Sun's motion can be neglected, and that planet and Sun move in circular orbits around their mutual centre of mass, the period is

$$P^2 = \frac{4\pi^2}{G(M+m)} a^3. \quad 9.2.2$$

Here  $a$  is the distance between Sun and planet.

*Exercise.* Express the period in terms of  $a_1$ , the radius of the planet's circular orbit around the centre of mass.

### 9.3 Kepler's Second Law from Conservation of Angular Momentum

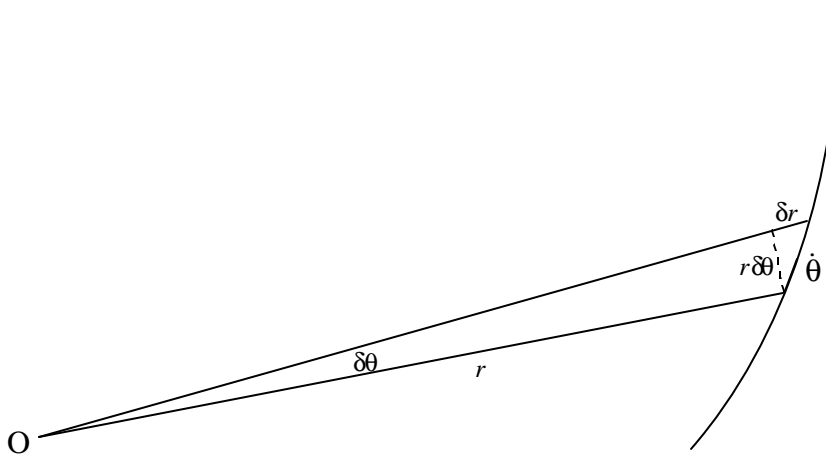


FIGURE IX.1

In figure IX.1, a particle of mass  $m$  is moving in some sort of trajectory in which the only force on it is directed towards or away from the point O. At some time, its polar coordinates are  $(r, \theta)$ . At a time  $\delta t$  later these coordinates have increased by  $\delta r$  and  $\delta\theta$ .

Using the formula one half base times height for the area of a triangle, we see that the area swept out by the radius vector is approximately

$$\delta A \approx \frac{1}{2} r^2 \delta \theta + \frac{1}{2} r \delta \theta \delta r. \quad 9.3.1$$

On dividing both sides by  $\delta t$  and taking the limit as  $\delta t \rightarrow 0$ , we see that the rate at which the radius vector sweeps out area is

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta}. \quad 9.3.2$$

But the angular momentum is  $mr^2\dot{\theta}$ , and since this is constant, the areal speed is also constant. The areal speed, in fact, is half the angular momentum per unit mass.

#### 9.4 *Some Functions of the Masses*

In section 9.5 I am going to consider the motion of two masses,  $M$  and  $m$  around their mutual centre of mass under the influence of their gravitational attraction. I shall probably want to make use of several functions of the masses, which I shall define here, as follows:

Total mass of the system:  $\mathbf{M} = M + m. \quad 9.4.1$

“Reduced mass”  $\mathbf{m} = \frac{Mm}{M+m}. \quad 9.4.2$

“Mass function”:  $\mathbb{M} = \frac{M^3}{(M+m)^2}. \quad 9.4.3$

No particular name:  $m_+ = m \left( 1 + \frac{m}{M} \right). \quad 9.4.4$

Mass ratio:  $q = m/M. \quad 9.4.5$

Mass fraction:  $\mu = m/(M+m). \quad 9.4.6$

The first four are of dimension  $\mathbf{M}$ ; the last two are dimensionless. When  $m \rightarrow 0$ ,  $\mathbf{m} \rightarrow m$ ,  $\mathbb{M} \rightarrow M$  and  $m_+ \rightarrow m$ .

(For those who may be interested, the fonts I have used are:

**M** Arial bold    **m** Schwaben Alt    *M* French script MT)

### 9.5 Kepler's First and Third Laws from Newton's Law of Gravitation

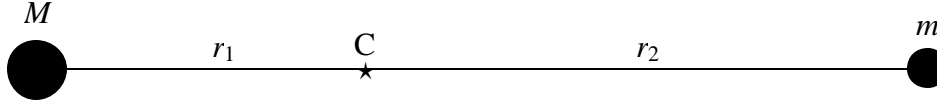


FIGURE IX.2

In figure IX.2 I illustrate two masses (they needn't be point masses – as long as they are spherically symmetric, they act gravitationally as if they were point masses) revolving about their common centre of mass C. At some time they are a distance  $r$  apart, where

$$r = r_1 + r_2, \quad r_1 = \frac{mr}{M+m}, \quad r_2 = \frac{Mr}{M+m} \quad 9.5.1$$

The equations of motion of  $m$  in polar coordinates (with C as pole) are

$$\text{Radial:} \quad \ddot{r}_2 - r_2 \dot{\theta}^2 = -GM/r^2. \quad 9.5.2$$

$$\text{Transverse:} \quad r_2 \ddot{\theta} + 2\dot{r}_2 \dot{\theta} = 0. \quad 9.5.3$$

Elimination of  $t$  between these equations will in principle give us the equation, in polar coordinates, of the path.

A slightly easier approach is to write down expressions for the angular momentum and the energy. The angular momentum per unit mass of  $m$  with respect to C is

$$h_2 = r_2^2 \dot{\theta}. \quad 9.5.4$$

The speed of  $m_2$  is  $\sqrt{\dot{r}_2^2 + r_2^2 \dot{\theta}^2}$ , and the speed of  $M$  is  $m/M$  times this. Some effort will be required of the reader to determine that the total energy  $E$  of the system is

$$E = \frac{1}{2} m_+ (\dot{r}_2^2 + r_2^2 \dot{\theta}^2) - \frac{GM\mu}{r_2}. \quad 9.5.5$$

[It is possible that you may have found this line quite difficult. The reason for the difficulty is that we are not making the approximation of a planet of negligible mass moving around a stationary Sun, but we are allowing both bodies to have comparable

masses and the move around their common centre of mass. You might first like to try the simpler problem of a planet of negligible mass moving around a stationary Sun. In that case  $r_1 = 0$  and  $r = r_2$  and  $\mathbf{m} \rightarrow m$ ,  $\mathbf{M} \rightarrow M$  and  $m_+ \rightarrow m$  .]

It is easy to eliminate the time between equations 9.5.4 and 9.5.5. Thus you can write

$\dot{r}_2 = \frac{dr_2}{dt} = \frac{dr_2}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta} \frac{dr_2}{d\theta}$  and then use equation 9.5.4 to eliminate  $\dot{\theta}$ . You should eventually obtain

$$\frac{m_+ h_2^2}{r_2^4} \left[ \left( \frac{dr_2}{d\theta} \right)^2 + r_2^2 \right] = 2E + \frac{2GM\mu}{r_2}. \quad 9.5.6$$

This is the differential equation, in polar coordinates, for the path of  $m$ . All that us now required is to integrate it to obtain  $r_2$  as a function of  $\theta$ .

At first, integration looks hopelessly difficult, but it proceeds by making one tentative substitution after another to see if we can't make it look a little easier. For example, we have (if we multiply out the square bracket)  $r_2$  in the denominator three times in the equation. Let's at least try the substitution  $w = 1/r_2$ . That will surely make it look a little easier. You will have to use  $\frac{dr_2}{d\theta} = \frac{dr_2}{dw} \frac{dw}{d\theta} = -\frac{1}{w^2} \frac{dw}{d\theta}$ , and after a little algebra, you should obtain

$$\left( \frac{dw}{d\theta} \right)^2 + \left( w - \frac{GM\mu}{m_+ h_2^2} \right)^2 = \frac{2E}{m_+ h_2^2} + \frac{G^2 M^2 \mu^2}{m_+^2 h_2^4}. \quad 9.5.7$$

This may at first sight not look like much of an improvement, but the right hand side is just a lot of constants, and, since it is positive, let's call the right hand side  $H^2$ . (In case you doubt that the right hand side is positive, the left hand side certainly is!) Also, make the obvious substitution

$$u = w - \frac{GM\mu}{m_+ h_2^2}, \quad 9.5.8$$

and the equation becomes almost trivial:

$$\left( \frac{du}{d\theta} \right)^2 + u^2 = H^2, \quad 9.5.9$$

from which we proceed to

$$\int d\theta = \pm \int \frac{du}{\sqrt{H^2 - u^2}}. \quad 9.5.10$$

At this stage you can choose either the + or the – and you can choose to make the next substitution  $u = H \sin \phi$  or  $u = H \cos \phi$ ; you'll get the same result in the end. I'll choose the plus sign and I'll let  $u = H \cos \phi$ , and I get  $\mathcal{U}\theta = -\mathcal{U}\phi$  and hence

$$\theta = -\phi + \omega, \quad 9.5.11$$

where  $\omega$  is the arbitrary constant of integration. Now you have to go back and remember what  $\phi$  was, what  $u$  was and what  $w$  was and what  $H$  was. Thus  $\theta - \omega = -\phi$ ,  $\therefore \cos(\theta - \omega) = \cos(-\phi) = \cos \phi = u/H = \dots$ and so on. Your aim is to get it in the form  $r_2 =$  function of  $\theta$ , and, if you persist, you should eventually get

$$r_2 = \frac{m_+ h_2^2 / (GM\mu)}{1 + \left(1 + \frac{2Eh_2^2 m_+}{G^2 M^2 \mu^2}\right)^{1/2} \cos(\theta - \omega)}. \quad 9.5.12$$

You'll immediately recognize this from equation 2.3.37 or 2.4.16 or 2.5.18:

$$r = \frac{l}{1 + e \cos(\theta - \omega)} \quad 2.3.37$$

as being the polar equation to a conic section (ellipse, parabola or hyperbola). Equation 9.5.12 is the equation of the path of the mass  $m$  *about the centre of mass* of the two bodies. The eccentricity is

$$e = \left(1 + \frac{2Eh_2^2 m_+}{G^2 M^2 \mu^2}\right)^{1/2}, \quad 9.5.13$$

or, if you now recall what are meant by  $\mu$  and  $m_+$ ,

$$e = \left(1 + \frac{2Eh_2^2}{G^2} \cdot \frac{(M+m)^3}{M^5 m}\right)^{1/2}. \quad 9.5.14$$

(Check the dimensions of this!)

*The eccentricity is less than 1, equal to 1, or greater than 1 (i.e. the path is an ellipse, a parabola or a hyperbola) according to whether the total energy  $E$  is negative, zero or positive.*

The semi latus rectum of the path of  $m$  relative to the centre of mass is of length

$$l_2 = \frac{m_+ h_2^2}{GM\mu}, \quad 9.5.15$$

or, if you now recall what are meant by  $m_+$  and  $\mu$  (see equations 9.3.4 and 9.3.6),

$$l_2 = \frac{h_2^2}{G} \cdot \frac{(M+m)^2}{M^3}. \quad 9.5.16$$

(Check the dimensions of this!)

We can also write equations 9.4.16 or 9.4.17 as

$$h_2^2 = GM l_2. \quad 9.5.17$$

At this point it is useful to recall what we mean by  $M$  and by  $h_2$ .  $M$  is the mass function, given by equation 9.4.3:

$$M = \frac{M^3}{(M+m)^2}. \quad 9.4.3$$

Let us suppose that the total energy is negative, so that the orbits are elliptical. The two masses are revolving in similar elliptic orbits around the centre of masses; the semi latus rectum of the orbit of  $m$  is  $l_2$ , and the semi latus rectum of the orbit of  $M$  is  $l_1$ , where

$$\frac{l_2}{l_1} = \frac{M}{m}. \quad 9.5.18$$

*Relative to  $M$*  the mass  $m$  is revolving in a larger but still similar ellipse with semi latus rectum  $l$  given by

$$l = \frac{M+m}{M} l_2. \quad 9.5.19$$

I am now going to define  $h$  as the *angular momentum per unit mass of  $m$  relative to  $M$* . In other words, we are working in a frame in which  $M$  is stationary and  $m$  is moving around  $M$  in an elliptic orbit of semi latus rectum  $l$ . Now angular momentum per unit mass is proportional to the areal speed, and therefore it is proportional to the square of the semilatus rectum. Thus we have

$$\frac{h}{h_2} = \left( \frac{l}{l_2} \right)^2 = \left( \frac{M+m}{M} \right)^2 \quad 9.5.20$$

Combining equations 9.5.18, 9.4.3, 9.5.19, 9.5.20 and 9.4.1 we obtain

$$h^2 = G\mathbf{M}l, \quad 9.5.21$$

where  $\mathbf{M}$  is the total mass of the system.

Once again:

The angular momentum per unit mass of  $m$  relative to the centre of mass is  $\sqrt{G\mathbf{M}l_2}$ , where  $l_2$  is the semi latus rectum of the orbit of  $m$  relative to the centre of mass, and it is  $\sqrt{G\mathbf{M}l}$  relative to  $M$ , where  $l$  is the semi latus rectum of the orbit of  $m$  relative to  $M$ .

If you were to start this analysis with the assumption that  $m \ll M$ , and that  $M$  remains stationary, and that the centre of mass coincides with  $M$ , you would find that either equation 9.5.17 or 9.5.21 reduces to

$$h^2 = G M l. \quad 9.5.22$$

The period of the elliptic orbit is area  $\div$  areal speed. The area of an ellipse is  $\pi ab = \pi a^2 \sqrt{1-e^2}$ , and the areal speed is half the angular momentum per unit mass (see section 9.3)  $= \frac{1}{2}h = \frac{1}{2}\sqrt{G\mathbf{M}l} = \frac{1}{2}\sqrt{G\mathbf{M}a(1-e^2)}$ . Therefore the period is  $P = \frac{2\pi}{G\mathbf{M}}a^{3/2}$ , or

$$P^2 = \frac{4\pi^2}{G\mathbf{M}}a^3, \quad 9.5.23$$

which is Kepler's third law.

We might also, while we are at it, express the eccentricity (equation) in terms of  $h$  rather than  $h_2$ , using equation 9.5.20. We obtain:

$$e = \left(1 + \frac{2Eh^2}{G^2 M m (M + m)}\right)^{1/2}. \quad 9.5.24$$

If we now substitute for  $h^2$  from equation 9.5.21, and invert equation 9.5.24, we obtain, for the energy of the system

$$E = \frac{Gm(M+m)(e^2-1)}{2l}, \quad 9.5.25$$

or for the energy or the system per unit mass of  $m$ :



$$E = \frac{G\mathbf{M}(e^2 - 1)}{2l}. \quad 9.5.26$$

Here  $\mathbf{M}$  is the mass of the system – i.e.  $M+m$ .  $E$  in equation 9.5.25 is the total energy of the system, which includes the kinetic energy of both masses as well as the mutual potential energy of the two, while  $E$  in equation 9.5.26 is merely  $E/m$ . That is, it is, as stated, the energy of the system per unit mass of  $m$ .

Equations 9.5.21 and 9.5.26 apply to any conic section. For the different types of conic section they can be written:

For an ellipse:

$$h = \sqrt{G\mathbf{M}a(1-e^2)}, \quad E = -\frac{G\mathbf{M}}{2a} \quad 9.5.27a,b$$

For a parabola:

$$h = \sqrt{2G\mathbf{M}q}, \quad E = 0 \quad 9.5.28a,b$$

For a hyperbola:

$$h = \sqrt{G\mathbf{M}a(e^2 - 1)}, \quad E = +\frac{G\mathbf{M}}{2a} \quad 9.5.29a,b$$

We see that the energy of an elliptic orbit is determined by the semi major axis, whereas the angular momentum is determined by the semi major axis and by the eccentricity. For a given semi major axis, the angular momentum is greatest when the orbit is circular.

Still referring the orbit of  $m$  with respect to  $M$ , we can find the speed  $V$  of  $m$  by noting that

$$E = \frac{1}{2}V^2 - \frac{G\mathbf{M}}{r}, \quad 9.5.30$$

and, by making use of the b-parts of equations 9.5.27-29, we find the following relations between speed of  $m$  in an orbit versus distance from  $M$ :

$$\text{Ellipse:} \quad V^2 = G\mathbf{M}\left(\frac{2}{r} - \frac{1}{a}\right). \quad 9.5.31$$

$$\text{Parabola:} \quad V^2 = \frac{2G\mathbf{M}}{r}. \quad 9.5.32$$

Hyperbola: 
$$V^2 = G\mathbf{M}\left(\frac{2}{r} + \frac{1}{a}\right). \quad 9.5.33$$

Circle: 
$$V^2 = \frac{G\mathbf{M}}{a}. \quad 9.5.34$$

Exercise: Show that in an elliptic orbit, the speeds at perihelion and aphelion are, respectively,  $\sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$  and  $\sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$  and that the ratio of perihelion speed to

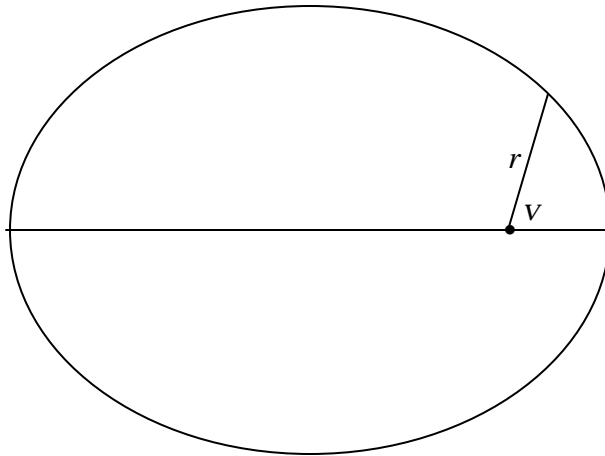
aphelion speed is, therefore,  $\frac{1+e}{1-e}$ . Take ten marks off if you incorrectly use the word velocity when you mean speed.

It might be noted at this point, from the definition of the astronomical unit (Chapter 8, section 8.1), that if distances are expressed in astronomical units, periods and time intervals in sidereal years,  $GM$  (where  $M$  is the mass of the Sun) has the value  $4\pi^2$ . The mass of a comet or asteroid is much smaller than the mass of the Sun, so that  $\mathbf{M} = M + m \simeq M$ . Thus, using these units, and to this approximation, equation 9.5.23 becomes merely  $P^2 = a^3$ .

### 9.6 Position in an Elliptic Orbit

The reader might like to refer back to Chapter 2, section 2.3, especially the part that deals with the polar equation to an ellipse, to be reminded of the meanings of the angles  $\theta$ ,  $\omega$  and  $v$ , which, in an astronomical context, are called, respectively, the *argument of latitude*, the *argument of perihelion* and the *true anomaly*. In this section I shall choose the initial line of polar coordinates to coincide with the major axis of the ellipse, so that  $\omega$  is zero and  $\theta = v$ . The equation to the ellipse is then

FIGURE IX.3



$$r = \frac{l}{1 + e \cos v}. \quad 9.6.1$$

I'll suppose that a planet is at perihelion at time  $t = T$ , and the aim of this section will be to find  $v$  as a function of  $t$ . The semi major axis of the ellipse is  $a$ , related to the semi latus rectum by

$$l = a(1 - e^2) \quad 9.6.2$$

and the period is given by

$$P^2 = \frac{4\pi^2}{G\mathbf{M}} a^3. \quad 9.6.3$$

Here the planet, of mass  $m$  is supposed to be in orbit around the Sun of mass  $M$ , and the origin, or pole, of the polar coordinates described by equation 9.6.1 is the Sun, rather than the centre of mass of the system. As usual,  $\mathbf{M} = M + m$ .

The radius vector from Sun to planet does not move at constant speed (indeed Kepler's second law states how it moves), but we can say that, over a complete orbit, it moves at an *average* angular speed of  $2\pi/P$ . The angle  $\frac{2\pi}{P}(t-T)$  is called the *mean anomaly* of the planet at a time  $t - T$  after perihelion passage. It is generally denoted by the letter  $M$ , which is already overworked in this chapter for various masses and functions of the masses. For mean anomaly, I'll try Copperplate Gothic Bold italic font,  $M$ . Thus

$$M = \frac{2\pi}{P}(t-T). \quad 9.6.4$$

The first step in our effort to find  $v$  as a function of  $t$  is to calculate the *eccentric anomaly*  $E$  from the mean anomaly. This was defined in figure II.11 of Chapter 2, and it is reproduced below.

In time  $t - T$ , the area swept out by the radius vector is the area FPB, and, because the radius vector sweeps out equal areas in equal times, this area is equal to the fraction  $(t-T)/P$  of the area of the ellipse. In other words, this area is  $\frac{(t-T)\pi ab}{P}$ . Now look at the area FP'B. Every ordinate of that area is equal to  $b/a$  times the corresponding ordinate of FPB, and therefore the area of FP'B is  $\frac{(t-T)\pi a^2}{P}$ . The area FP'B is also equal to the sector OP'B minus the triangle OP'F. The area of the sector OP'B is  $\frac{E}{2\pi} \times \pi a^2 = \frac{1}{2} E a^2$ , and the area of the triangle OP'F is  $\frac{1}{2} a e \times a \sin E = \frac{1}{2} a^2 e \sin E$ .

$$\therefore \frac{(t-T)\pi a^2}{P} = \frac{1}{2} E a^2 - \frac{1}{2} a^2 e \sin E.$$

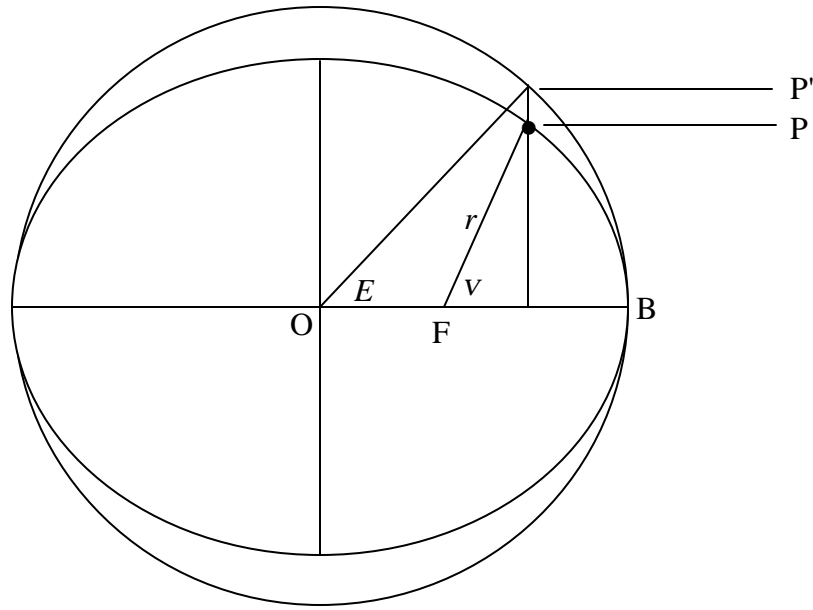


FIGURE II.11

Multiply both sides by  $2/a^2$ , and recall equation 9.6.4, and we arrive at the required relation between the mean anomaly  $M$  and the eccentric anomaly  $E$ :

$$M = E - e \sin E. \quad 9.6.5$$

This is *Kepler's equation*.

The first step, then, is to calculate the mean anomaly  $M$  from equation 9.6.4, and then calculate the eccentric anomaly  $E$  from equation 9.6.5. This is a transcendental equation, so I'll say a word or two about solving it in a moment, but let's press on for the time being. We now have to calculate the true anomaly  $v$  from the eccentric anomaly. This is done from the geometry of the ellipse, with no dynamics, and the relation is given in Chapter 2, equations 2.3.16 and 2.3.17c, which are reproduced here:

$$\cos v = \frac{\cos E - e}{1 - e \cos E}. \quad 2.3.16$$

From trigonometric identities, this can also be written

$$\sin v = \frac{\sqrt{1-e^2} \sin E}{1-e \cos E}, \quad 2.3.17a$$

or

$$\tan v = \frac{\sqrt{1-e^2} \sin E}{\cos E - e} \quad 2.3.17b$$

or

$$\tan \frac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2}E. \quad 2.3.17c$$

If we can just solve equation 9.6.5 (Kepler's equation), we shall have done what we want – namely, find the true anomaly as a function of the time.

The solution of Kepler's equation is in fact very easy. We write it as

$$f(E) = E - e \sin E - M \quad 9.6.6$$

from which

$$f'(E) = 1 - e \cos E, \quad 9.6.7$$

and then, by the usual Newton-Raphson process:

$$E = \frac{M - e(E \cos E - \sin E)}{1 - e \cos E}. \quad 9.6.8$$

The computation is then extraordinarily rapid (especially if you store  $\cos E$  and don't calculate it twice!).

*Example:*

Suppose  $e = 0.95$  and that  $M = 245^\circ$ . Calculate  $E$ . Since the eccentricity is very large, one might expect the convergence to be slow, and also  $E$  is likely to be very different from  $M$ , so it is not easy to make a first guess for  $E$ . You might as well try  $245^\circ$  for a first guess for  $E$ . You should find that it converges to ten significant figures in a mere four iterations. Even if you make a mindlessly stupid first guess of  $E = 0^\circ$ , it converges to ten significant figures in only nine iterations.

There are a few exceptional occasions, hardly ever encountered in practice, and only for eccentricities greater than about 0.99, when the Newton-Raphson method will not converge when you make your first guess for  $E$  equal to  $M$ . Charles and Tatum (*CelestialMechanics and Dynamical Astronomy* **69**, 357 (1998)) have shown that the Newton-Raphson method will *always* converge if you make your first guess  $E = \pi$ . Nevertheless, the situations where Newton-Raphson will not converge with a first guess of  $E = M$  are unlikely to be encountered except in almost parabolic orbits, and usually a first guess of  $E = M$  is faster than a first guess of  $E = \pi$ . The chaotic behaviour

of Kepler's equation on these exceptional occasions is discussed in the above paper and also by Stumpf (*Cel. Mechs. and Dyn. Astron.* **74**, 95 (1999)) and references therein.

*Exercise:* Show that a good first guess for  $E$  is

$$E = M + x(1 - \frac{1}{2}x^2), \quad 9.6.9$$

where 
$$x = \frac{e \sin M}{1 - e \cos M}. \quad 9.6.10$$

*Exercise:* Write a computer program in the language of your choice for solving Kepler's equation. The program should accept  $e$  and  $M$  as input, and return  $E$  as output. The Newton-Raphson iteration should be terminated when  $|(E_{\text{new}} - E_{\text{old}})/E_{\text{old}}|$  is less than some small fraction to be determined by you.

*Exercise:* An asteroid is moving in an elliptic orbit of semi major axis 3 AU and eccentricity 0.6. It is at perihelion at time = 0. Calculate its distance from the Sun and its true anomaly one sidereal year later. You may take the mass of the asteroid and the mass of Earth to be negligible compared with the mass of the Sun. In that case, equation 9.6.3 is merely

$$P^2 = \frac{4\pi^2}{GM} a^3,$$

where  $M$  is the mass of the Sun, and, if  $P$  is expressed in sidereal years and  $a$  in AU, this becomes just  $P^2 = a^3$ . Thus you can immediately calculate the period in years and hence, from equation 9.5.4 you can find the mean anomaly. From there, you have to solve Kepler's equation to get the eccentric anomaly, and the true anomaly from equation 2.3.16 or 17. Just make sure that you get the quadrant right.

*Exercise:* Write a computer program that will give you the true anomaly and heliocentric distance as a function of time since perihelion passage for an asteroid whose elliptic orbit is characterized by  $a$ ,  $e$ . Run the program for the asteroid of the previous exercise for every day for a complete period.

You are now making some real progress towards ephemeris computation!

## 9.7 Position in a Parabolic Orbit

When a "long-period" comet comes in from the Oort belt, it typically comes in on a highly eccentric orbit, of which we can observe only a very short arc. Consequently, it is often impossible to determine the period or semi major axis with any degree of reliability or to distinguish the orbit from a parabola. There is therefore frequent occasion to have to understand the dynamics of a parabolic orbit.

We have no mean or eccentric anomalies. We must try to get  $v$  directly as a function of  $t$  without going through these intermediaries.

The angular momentum per unit mass is given by equation 9.5.28a:

$$h = r^2 \dot{v} = \sqrt{2GMq}, \quad 9.7.1$$

where  $v$  is the true anomaly and  $q$  is the perihelion distance.

But the equation to the parabola (see equation 2.4.16) is

$$r = \frac{2q}{1 + \cos v}, \quad 9.7.2$$

or (see section 3.8 of Chapter 3), by making use of the identity

$$\cos v = \frac{1 - u^2}{1 + u^2}, \quad \text{where} \quad u = \tan \frac{1}{2}v, \quad 9.7.3a,b$$

the equation to the parabola can be written

$$r = q \sec^2 \frac{1}{2}v. \quad 9.7.4$$

Thus, by substitution of equation 9.7.4 into 9.7.1 and integrating, we obtain

$$q^2 \int_0^v \sec^4 \left( \frac{1}{2}v \right) dv = \sqrt{2GMq} \int_T^t dt. \quad 9.7.5$$

Upon integration (drop me an email if you get stuck!) this becomes

$$u + \frac{1}{3}u^3 = \frac{\sqrt{\frac{1}{2}GM}}{q^{3/2}}(t - T). \quad 9.7.6$$

This equation, when solved for  $u$ , gives us  $v$  as a function of  $t$ . As explained at the end of section 9.5, if  $q$  is in astronomical units and  $t - T$  is in sidereal years, and if the mass of the comet is negligible compared with the mass of the Sun, this becomes

$$u + \frac{1}{3}u^3 = \frac{\pi\sqrt{2}(t - T)}{q^{3/2}} \quad 9.7.7$$

$$\text{or} \quad 3u + u^3 - C = 0, \quad \text{where} \quad C = \frac{\pi\sqrt{18}(t - T)}{q^{3/2}}. \quad 9.7.8a,b$$

There is a choice of methods available for solving equation 9.7.8, so it might be that the only difficulty is to decide which of the several methods you want to use!

*Method 1:* Just solve it by Newton-Raphson iteration. Thus  $f = 3u + u^3 - C = 0$  and  $f' = 3(1 + u^2)$ , so that the Newton-Raphson  $u = u - f/f'$  becomes

$$u = \frac{2u^3 + C}{3(1 + u^2)}, \quad 9.7.9$$

which should converge quickly. For economy, calculate  $u^2$  only once per iteration.

*Method 2:*

$$\text{Let} \quad u = x - 1/x \quad \text{and} \quad C = c - 1/c. \quad 9.7.10a,b$$

Then equation 9.7.8a becomes

$$x = c^{1/3}. \quad 9.7.11$$

Thus, as soon as  $c$  is found,  $x$ ,  $u$  and  $v$  can be calculated from equations 9.7.11, 10a, and 3a or b, and the problem is finished – as soon as  $c$  is found!

So, how do we find  $c$ ? We have to solve equation 9.7.10b.

*Method 2a:*

Equation 9.7.10b can be written as a quadratic equation:

$$c^2 - Cc - 1 = 0. \quad 9.7.12$$

Just be careful that you choose the correct root; you should end with  $v$  having the same sign as  $t - T$ .

*Method 2b:*

$$\text{Let} \quad C = 2 \cot 2\phi \quad 9.7.13$$

and calculate  $\phi$ . But by a trigonometric identity,

$$2 \cot 2\phi = \cot \phi - 1/\cot \phi \quad 9.7.14$$

so that, by comparison with equation 9.6.10b, we see that



$$c = \cot \phi . \quad 9.7.15$$

Again, just make sure that you choose the right quadrant in calculating  $\phi$  from equation 9.7.13, so as to be sure that you end with  $v$  having the same sign as  $t - T$ .

*Example:* Solve the equation  $3u + u^3 = 1.6$  by all three methods. (Methods 1, 2a, and 2b.)

*Example:* A comet is moving in an elliptic orbit with perihelion distance 0.9 AU. Calculate the true anomaly and heliocentric distance 20 days after perihelion passage. (A sidereal year is 365.25636 days.)

*Exercise:* Write a computer program that will return the true and anomaly as a function of time, given the perihelion distance of a parabolic orbit. Test it with your answer for the previous example.

## 9.8 Position in a Hyperbolic Orbit

If an interstellar comet were to encounter the solar system from interstellar space, it would pursue a hyperbolic orbit around the Sun. To date, no such comet with an original hyperbolic orbit has been found, although there is no particular reason why we might not find one some night. However, a comet with a near-parabolic orbit from the Oort belt may approach Jupiter on its way in to the inner solar system, and its orbit may be perturbed into a hyperbolic orbit. This will result in its ultimate loss from the solar system. Several examples of such cometary orbits are known. There is evidence, from radar studies of meteors, of meteoroidal dust encountering Earth at speeds that are hyperbolic with respect to the Sun, although whether these are on orbits that are originally hyperbolic (and are therefore from interstellar space) or whether they are of solar system origin and have been perturbed by Jupiter into hyperbolic orbits is not known.

I must admit to not having actually carried out a calculation for a hyperbolic orbit, but I think we can just proceed in a manner similar to an ellipse or a parabola. Thus we can start with the angular momentum per unit mass:

$$h = r^2 \dot{v} = \sqrt{GMl}, \quad 9.8.1$$

$$\text{where} \quad r = \frac{l}{1 + e \cos v} \quad 9.8.2$$

$$\text{and} \quad l = a(e^2 - 1). \quad 9.8.3$$

If we use astronomical units for distance and mass, we obtain

$$\int_0^v \frac{dv}{(1 + e \cos v)^2} = \frac{2\pi}{a^{3/2}(e^2 - 1)^{3/2}} \int_T^t dt . \quad 9.8.4$$

Here I am using astronomical units of distance and mass and have therefore substituted  $4\pi^2$  for  $G M$ .

I'm going to write this as

$$\int_0^v \frac{dv}{(1 + \cos v)^2} = \frac{2\pi(t-T)}{a^{3/2}(e^2 - 1)^{3/2}} = \frac{Q}{(e^2 - 1)^{3/2}} , \quad 9.8.5$$

where  $Q = \frac{2\pi(t-T)}{a^{3/2}}$ . Now we have to integrate this.

*Method 1.*

Guided by the elliptical case, but bearing in mind that we are now dealing with a hyperbola, I'm going to try the substitution

$$\cos v = \frac{e - \cosh E}{e \cosh E - 1} . \quad 9.8.6$$

If you try this, I think you'll end up with

$$e \sinh E - E = Q . \quad 9.8.7$$

This is just the analogy of Kepler's equation.

The procedure, then, would be to calculate  $Q$  from equation 9.8.5. Then calculate  $E$  from equation 9.8.7. This could be done, for example, by a Newton-Raphson iteration in quite the same way as was done for Kepler's equation in the elliptic case, the iteration now taking the form

$$E = \frac{Q + e(E \cosh E - \sinh E)}{e \cosh E - 1} . \quad 9.8.8$$

Then  $v$  is found from equation 9.8.6, and the heliocentric distance is found from the polar equation to a hyperbola:

$$r = \frac{a(e^2 - 1)}{1 + e \cos v}. \quad 9.8.9$$

*Method 2.*

Method 1 should work all right, but it has the disadvantage that you may not be as familiar with sinh and cosh as you are with sin and cos, or there may not be a sinh or cosh button your calculator. I believe there are SINH and COSH functions in FORTRAN, and there may well be in other computing languages. Try it and see. But maybe we'd like to try to avoid hyperbolic functions, so let's try the brilliant substitution

$$\cos v = -\frac{u(u - 2e) + 1}{u(eu - 2) + e}. \quad 9.8.10$$

You may have noticed, when you were learning calculus, that often the professor would make a brilliant substitution, and you could see that it worked, but you could never understand what made the professor think of the substitution. I don't want to tell you what made me think of this substitution, because, when I do, you'll see that it isn't really very brilliant at all. I remembered that

$$\cosh E = \frac{1}{2}(e^E + e^{-E}) \quad 9.8.11$$

and then I let  $e^E = u$ , so

$$\cosh E = \frac{1}{2}(u + 1/u), \quad 9.8.12$$

and I just substituted this into equation 9.8.6 and I got equation 9.8.10. Now if you put the expression 9.8.10 for  $\cos v$  into equation 9.8.5, you eventually, after a few lines, get something that you can integrate. Please do work through it. In the end, on integration of equation 9.8.5, you should get

$$\frac{1}{2}e(u - 1/u) - \ln u = Q. \quad 9.8.13$$

You already know from Chapter 1 how to solve the equation  $f(x) = 0$ , so there is no difficulty in solving equation 9.8.13 for  $u$ . Newton-Raphson iteration results in

$$u = \frac{2u[e - u(1 - Q \ln u)]}{u(eu - 2) + 2}, \quad 9.8.14$$

and this should converge in the usual rapid fashion.

So the procedure in method 2 is to calculate  $Q$  from equation 9.8.5, then calculate  $u$  from equation 9.8.14, and finally  $v$  from equation 9.8.10 – all very straightforward.

*Exercise:* Set yourself a problem to make sure that you can carry through the calculation. Then write a computer program that will generate  $v$  and  $r$  as a function of  $t$ .

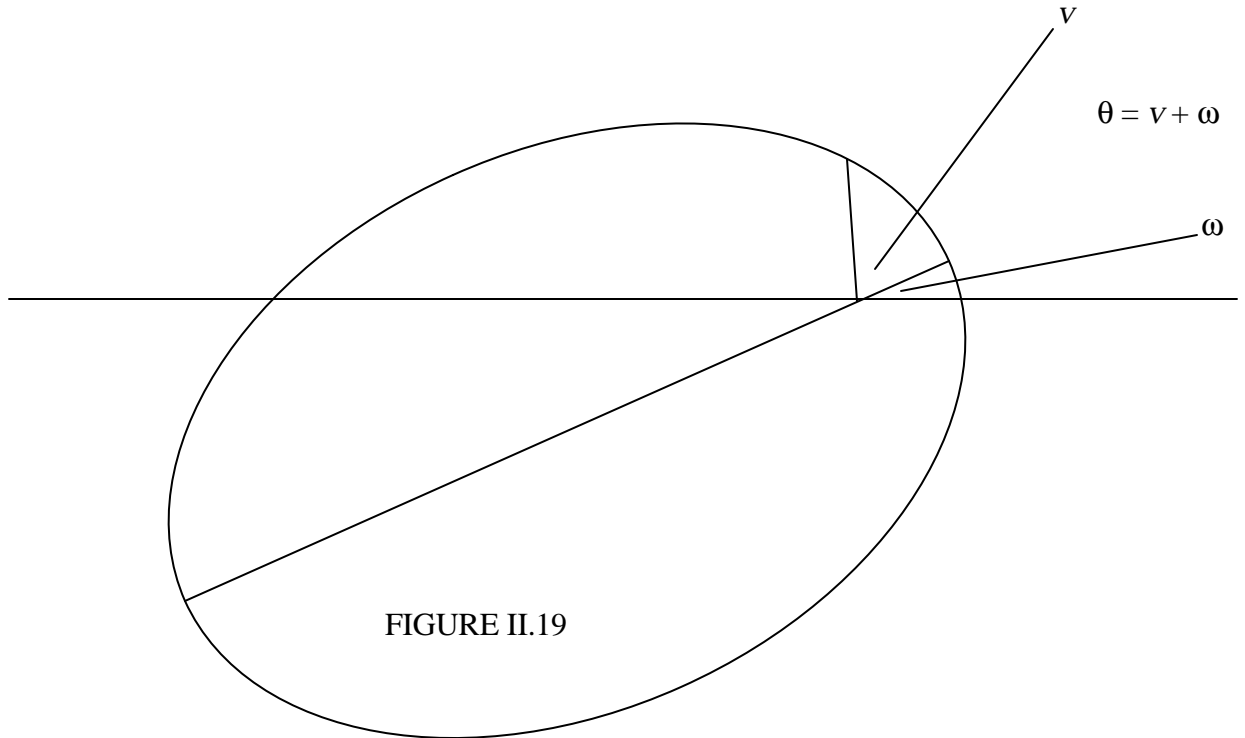
### 9.9 Orbital Elements and Velocity Vector

In two dimensions, an orbit can be completely specified by *four orbital elements*. Three of them give the size, shape and orientation of the orbit. They are, respectively,  $a$ ,  $e$  and  $\omega$ . We are familiar with the semi major axis  $a$  and the eccentricity  $e$ . The angle  $\omega$ , the argument of perihelion, was illustrated in figure II.19, which is reproduced here. It is the angle that the major axis makes with the initial line of the polar coordinates. Figure II.19 reminds us of the relation between the argument of perihelion  $\omega$ , the argument of latitude  $\theta$  and the true anomaly  $v$ . We remind ourselves here of the equation to a conic section

$$r = \frac{l}{1 + e \cos v} = \frac{l}{1 + e \cos(\theta - \omega)}, \quad 9.9.1$$

where the semi latus rectum  $l$  is  $a(1 - e^2)$  for an ellipse, and  $a(e^2 - 1)$  for a hyperbola. For a hyperbola, the parameter  $a$  is usually called the semi transverse axis. For a parabola, the size is generally described by the perihelion distance  $q$ , and  $l = 2q$ .

The fourth element is needed to give information about where the planet is in its orbit at a particular time. Usually this is  $T$ , the time of perihelion passage. In the case of a circular orbit this cannot be used. One could instead give the time when  $\theta = 0$ , or the value of  $\theta$  at some specified time.



Refer now to figure IX.4

We'll suppose that at some time  $t$  we know the coordinates  $(x, y)$  or  $(r, \theta)$  of the planet, and also the velocity – that is to say the speed and direction, or the  $x$ - and  $y$ - or the radial and transverse components of the velocity. That is, we know four quantities. The subsequent path of the planet is then determined. In other words, given the four quantities (two components of the position vector and two components of the velocity vector), we should be able to determine the four elements  $a, e, \omega$  and  $T$ . Let us try.

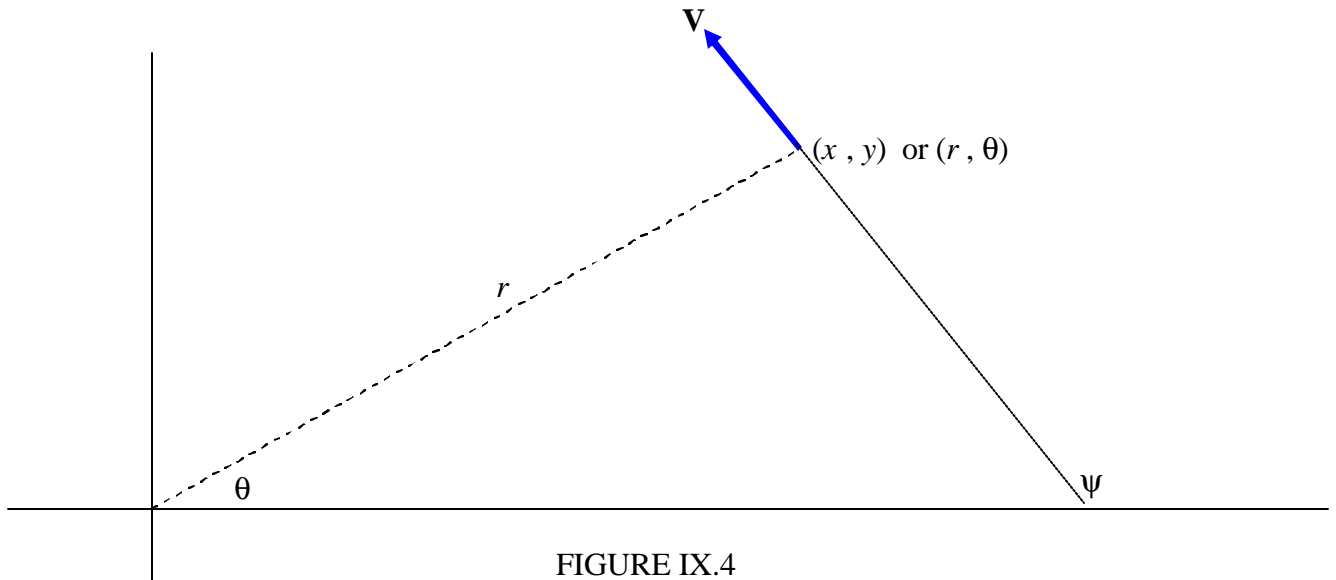


FIGURE IX.4

The semi major axis is easy. It's determined from equation 9.5.31:

$$V^2 = G\mathbf{M}\left(\frac{2}{r} - \frac{1}{a}\right). \quad 9.5.31$$

If distances are expressed in AU and if the speed is expressed in units of  $29.7846917 \text{ km s}^{-1}$ ,  $G \mathbf{M} = 1$ , so that the semi major axis in AU is given by

$$a = \frac{r}{2 - rV^2}. \quad 9.9.2$$

In other words, if we know the speed and the heliocentric distance, the semi major axis is known. If  $a$  turns out to be infinite – in other words, if  $V^2 = 2/r$ , the orbit is a parabola; and if  $a$  is negative, it is a hyperbola. For an ellipse, of course, the period in sidereal years is given by  $P^2 = a^3$ .

From the geometry of figure IX.4, the transverse component of  $\mathbf{V}$  is  $V \sin(\psi - \theta)$ , which is known, the magnitude and direction of  $\mathbf{V}$  being presumed known. Therefore the angular momentum per unit mass is  $r$  times this, and, for an elliptic orbit, this is related to  $a$  and  $e$  by equation 9.5.27a:

$$h = \sqrt{G \mathbf{M} a(1 - e^2)}. \quad 9.5.27a$$

$$\therefore r V \sin(\psi - \theta) = \sqrt{G \mathbf{M} a(1 - e^2)}. \quad 9.9.2$$

Again, if distances are expressed in AU and  $V$  in units of  $29.7846917 \text{ km s}^{-1}$ ,  $G \mathbf{M} = 1$ , and so

$$r V \sin(\psi - \theta) = \sqrt{a(1 - e^2)}. \quad 9.9.3$$

Thus  $e$  is determined.

The equation to an ellipse is

$$r = \frac{a(1 - e^2)}{1 + e \cos(\theta - \omega)}, \quad 9.9.4$$

so, provided the usual care is taken in choosing the quadrant,  $\omega$  is now known.

From there we proceed:

$$v = \theta - \omega, \quad \cos E = \frac{e + \cos v}{1 + e \cos v}, \quad M = E - e \sin E = \frac{2\pi}{P}(t - T),$$

and  $T$  is found. The procedure for a parabola or a hyperbola is similar.

*Example:*

At time  $t = 0$ , a comet is at  $x = +3.0$ ,  $y = +6.0$  AU and it has a velocity with components  $\dot{x} = -0.2$ ,  $\dot{y} = +0.4$  times  $29.7846917 \text{ km s}^{-1}$ . Find the orbital elements  $a$ ,  $e$ ,  $\omega$  and  $T$ .

*Solution:*

Note in what follows that, although I am quoting numbers to only a few significant figures, the calculation at all times carries all ten figures that my hand calculator allows. You will not get exactly the same results unless you do likewise. Do not prematurely round off. But please let me know if you think I have made any actual mistakes. I am using astronomical units of distance, sidereal years for time and speed in units of  $29.8 \text{ km s}^{-1}$ .

$$r = 6.708, \quad \theta = 63^\circ 26', \quad V = 0.4472, \quad \psi = 116^\circ 34'$$

Be sure to get the quadrants right!

$$\underline{a = 10.19 \text{ AU}} \quad P = 32.5 \quad \underline{e = 0.6593} \quad \cos(\theta - \omega) = -0.21439$$

And now we are faced with a dilemma.  $\theta - \omega = 102^\circ 23'$  or  $257^\circ 37'$ . Which is it? This is a typical “quadrant problem”, and it cannot be ignored. The two possible solutions give  $\omega = 321^\circ 03'$  or  $165^\circ 49'$ , and we have to decide which is correct.

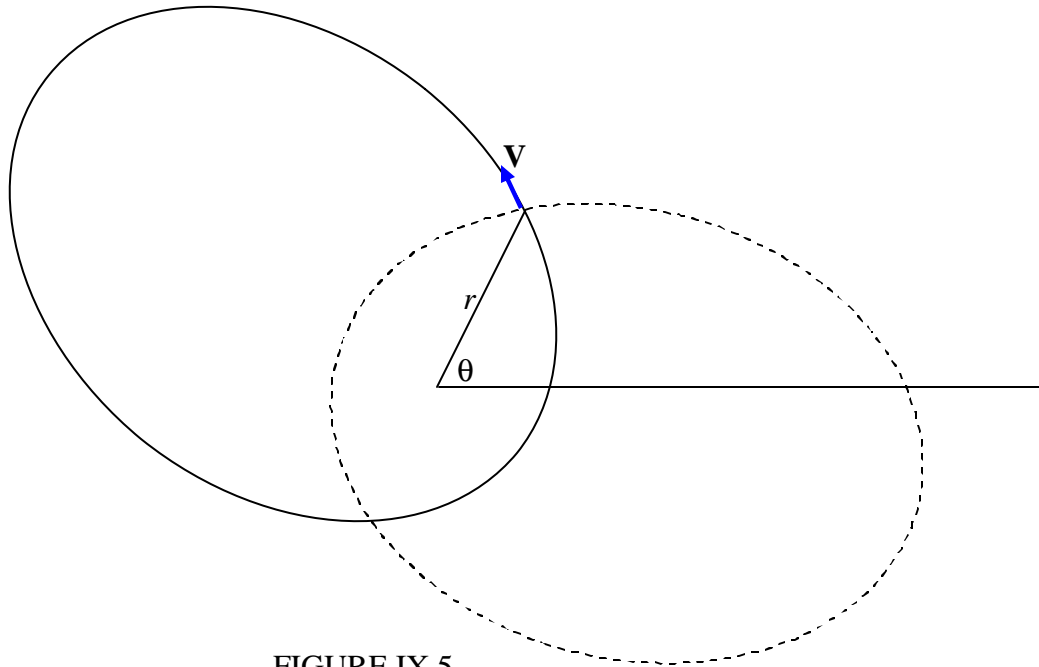


FIGURE IX.5

The two solutions are drawn in figure IX.5. The continuous curve is the ellipse for  $\omega = 321^\circ 03'$  and the dashed curve is the curve for  $\omega = 165^\circ 49'$ . I have also drawn in the velocity vector at  $(r, \theta)$ , and it is clear from the drawing that the continuous curve with  $\omega = 321^\circ 03'$  is the correct ellipse. Is there a way of deducing this from the equations rather than going to the trouble of drawing the ellipses? There probably is, and I shall leave it to you, the reader, to find it. Meanwhile, having found the correct ellipse by drawing, I am going to press on, having fulfilled the mission of pointing out that quadrant problems do occur in trigonometry, and they must, one way or another, be solved. We now have

$$\underline{\omega = 321^\circ 03'}$$

From this point we go:

$$v = 102^\circ 23', \quad \cos E = 0.51817,$$

and again we are presented with a dilemma, for this gives  $E = 58^\circ 47'$  or  $301^\circ 13'$ , and we have to decide which one is correct. From the geometrical meaning of  $v$  and  $E$ , we can understand that they are equal when each of them is either  $0^\circ$  or  $180^\circ$ . Since  $v < 180^\circ$ ,  $E$  must also be less than  $180^\circ$ , so the correct choice is  $E = 58^\circ 47' = 1.0261$  rad. From there, we have

$$M = 26^\circ 29' = 0.46218 \text{ rad}, \quad \underline{T = -2.392 \text{ sidereal years}},$$

and the elements are now completely determined.

*Problem:* Write a computer program, in the language of your choice, in which the input data are  $x, y, \dot{x}, \dot{y}$ , and the output is  $a, e, \omega$  and  $T$ . You will probably want to keep it simple at first, and deal only with ellipses. Therefore, if the program calculates that  $a$  is not positive, exit the program then. I'm not sure how you will solve the quadrant problems. That will be up to your ingenuity. Don't forget that many languages have an ARCTAN2 function. Later, you will want to expand the program and deal with *any* set of  $x, y, \dot{x}, \dot{y}$ , with a resulting orbit that may be any of the conic sections. Particularly annoying cases may be those in which the planet is heading straight for the Sun, with no transverse component of velocity, so that it is moving in a straight line, or a circular orbit, in which case  $T$  is undefined.

Notice that the problem we have dealt with in this section is the opposite of the problem we dealt with in sections 9.6, 9.7 and 9.8. In the latter, we were given the elements, and we calculated the position of the planet as a function of time. That is, we calculated an ephemeris. In the present section, we are given the position and velocity at some time and are asked to calculate the elements. Both problems are of comparable difficulty. Perhaps the latter is slightly easier than the former, since we don't have to solve Kepler's



equation. This might give the impression that calculating the orbital elements of a planet is of comparable difficulty to, or even slightly easier than, calculating an ephemeris from the elements. This is, in practice, very far from the case, and in fact calculating the elements from the observations is *very* much more difficult than generating an ephemeris. In this section, we have calculated the elements, *given the position and velocity vectors*. In real life, when a new planet swims into our ken, we have no idea of the distance or of the speed or the direction of motion. All we have is a set of positions against the starry background, and the most difficult part of the problem of determining the elements is to determine the distance.

The next chapter will deal with generating an ephemeris (right ascension and declination as a function of time) from the orbital elements in the real three-dimensional situation. Calculating the elements from the observations will come much later.

### 9.10 *Osculating Elements*

We have seen that, if we know the position and velocity vectors at a particular instant of time, we can calculate the orbital elements of a planet with respect to the Sun. If the Sun and one planet (or asteroid or comet) are the only two bodies involved, and if the Sun is spherically symmetric and if we can ignore the refinements of general relativity, the planet will pursue that orbit indefinitely. In practice, however, the orbit is subject to *perturbations*. In the case of most planets moving around the Sun, the perturbations are caused mostly by the gravitational attractions of the other planets. For Mercury, the refinements of general relativity are important. The asphericity of the Sun is unimportant, although for satellites in orbit around aspherical planets, the asphericity of the planet becomes important. In any case, for one reason or another, in practice, an orbit is subject to perturbations, and the planet does not move indefinitely in the orbit that is calculated from the position and velocity vectors at a particular time. The orbit that is calculated from the position and velocity vectors at a particular instant of time is called the *osculating orbit*, and the corresponding orbital elements are the *osculating elements*. The instant of time at which the position and velocity vectors are specified is the *epoch of osculation*. The osculating orbit touches (“kisses”) the real, perturbed orbit at the epoch of osculation. The verb “to osculate”, from the Latin *osculare*, means “to kiss”.

For the time being, then, we shall be satisfied with calculating an *osculating orbit*, and with generating an ephemeris from the *osculating* elements. In computing practice, for asteroid work, people compute elements for an epoch of osculation that is announced by and changed by the Minor Planet Center of the International Astronomical Union every 200 days.

### 9.11 *Mean Distance in an Elliptic Orbit*

It is sometimes said that “ $a$ ” in an elliptic orbit is the “mean distance” of a planet from the Sun. In fact  $a$  is the semi major axis of the orbit. Whether and in what sense it might also be the “mean distance” is worth a moment of thought.

It was the late Professor C. E. M Joad whose familiar answer to the weighty questions of the day was “It all depends what you mean by...” And the “mean distance” depends on whether you mean the distance averaged over the true anomaly  $v$  or over the time. The mean distance averaged over the true anomaly is  $\frac{1}{\pi} \int_0^\pi r dv$ , where  $r = l/(1 + e \cos v)$ . If you are looking for some nice substitution to help you to integrate this, equation 2.13.6 does very nicely, and you soon find the unexpected result that the mean distance, averaged over the mean anomaly, is  $b$ , the semi *minor* axis.

On the other hand, the mean distance averaged over the time is  $\frac{1}{2} P \int_0^{1/P} r dt$ . This one is slightly more tricky, but, following the hint for evaluating  $\frac{1}{\pi} \int_0^\pi r dv$ , you could try expressing  $r$  and  $v$  in terms of the eccentric anomaly. It will take you a moment or so, but you should eventually find that the mean distance averaged over the time is  $a(1 + \frac{1}{2}e^2)$ .

It is often pointed out that, because of Kepler’s second law, a planet spends more time far from the Sun than it does near to the Sun, which is why we have longer summers than winters in the northern hemisphere. An easy exercise would be to ask you what fraction of its orbital period does a planet spend on the sunny side of a latus rectum. A slightly more difficult exercise would be to ask: What fraction of its orbital period does a planet spend closer to the Sun than its mean (time-averaged) distance? You’d first have to ask, what is the true anomaly when  $r = a(1 + \frac{1}{2}e^2)$ ? Then you need to calculate the fraction of the area of the orbit. Area in polar coordinates is  $\frac{1}{2} \int r^2 dv$ . I haven’t tried this, but, if it proves difficult, I’d try and write  $r$  and  $v$  in terms of the eccentric anomaly  $E$  and see if that helped.