The Einstein Field Equations

Chris Ormel*
Kapteyn Instituut
Landleven 12, 9747 AD Groningen, The Netherlands

November 20, 2001

*ormel@astro.rug.nl

Contents

Introduction

The Einstein equations are often refered to, even in informal (i.e. unscientific) conversations as the (most) fundamental laws governing the structure and predicting the future of the universe. However for most people this subject still remains schrouded in mystery, even for some undergraduate students. My purpose is to unravel the Einstein equations, to clarify its components and the relations they describe. In the last section I will derive symbols, used in cosmology from the Einstein equations. It is not the intention to give a formal derivation, but more an intuitive feeling for the ideas behind this theory. However, most students should be able to understand the math.

prerequistites, notation

In order to be able to follow this paper, one should be equipped with a fundamental set of 'tensor analysis'. The most important properties being:

- 4-vectors e.g. $x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x})$[1]
- 4-tensor like $T^{\mu\nu}$ (16 components) or $R^\alpha{}_{\beta\gamma\delta}$ (256 components)
- contravariant ($x^\mu,\ \partial^\mu$) and covariant ($x_\mu,\ \partial_\mu$) components of vectors and (in general) tensors or derivatives.
- Using the metric tensor to raise or lower indices: $V_\mu = g_{\mu\nu} V^\nu$
- contraction, which reduces the rank of a tensor with two, e.g. $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \sum_{\lambda=0}^{3} R^\lambda{}_{\mu\lambda\nu}$
  Here the λ's are *dummy indices* and could be replaced by any other greek letter.
- tensor transformation from one frame to another:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \qquad A'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} A^{\alpha\beta}$$

In this paper I will stick to the following notation

- greek letters for 4-vectors and latin indices for (spatial) 3-vectors.
- ∂_μ for (ordinary) differentiation $\frac{\partial}{\partial x^\mu}$, ∇_μ for the covariant derivative [2]

[1] the speed of light in x^0 (ct) is to be taken 1 in this paper

[2] what exactly the covariant derivative is, will be clear after reading this paper

1 The Einstein Field Equations

The Einstein Field equations are:

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{1}$$

In which

- $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, which measures the curvature of spacetime
- G is Newtons constant[3]
- $T_{\mu\nu}$ is the energy-momentum tensor.
- $g_{\mu\nu}$ is the metric, a generalisation of the Minkowski metric $\eta_{\mu\nu}$
- $R_{\mu\nu}$ is the Ricci tensor, a contraction of the Riemann curvature tensor.

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{2}$$

- R is the curvature scaler, the contraction of the Ricci tensor

$$R = R^{\mu}{}_{\mu} \tag{3}$$

In the following 4 chapters I will describe the varies components of the Einstein equations. The last chapter is an application of the Einstein equations on cosmology.

2 The metric $g_{\mu\nu}$

In special relativity we have the invariant

$$d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{4}$$

where $\eta_{\alpha\beta} =$ diag $(1, -1, -1, -1)$ is the *Minkowski metric*. The Minkowski metric only applies in special relativity, where we have $\frac{d^2\xi^{\alpha}}{d\tau^2} = 0$.

According to the *principle of relativity* there is always a freely falling (comoving) coordinate frame ξ^{α} in which $\frac{d^2\xi^{\alpha}}{d\tau^2}$ vanishes. In another coordinate frame x^{μ} the invariant $d\tau^2$ becomes

$$d\tau^2 = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} dx^{\mu} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{5}$$

Where $g_{\mu\nu}$ is the *metric tensor*

$$g_{\mu\nu} = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} \tag{6}$$

[3] In contrary to c, I will not set this constant 1

3 The Energy-Momentum Tensor

The energy-momentum tensor $T_{\mu\nu}$ is a symmetric tensor, which conserves energy and momentum. It's described by this conservation law, rather than an exact defenition. I will give an example of this tensor for fluids.
Consider a fluid characterised by the density field $\rho(\mathbf{x})$ and four-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx^k}{d\tau}\right) = (\gamma, \gamma\mathbf{v}) \tag{7}$$

The boost term γ restricts the discussion to special relativity, but our goal is to show the conservation of energy and momentum. Let's define the momentum-energy tensor to be [4]

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{8}$$

Where $g^{\mu\nu}$ will be replaced by $\eta^{\mu\nu}$.

Now in a Newtonian case ($v << c$), the pressure (p) is small compared to the density of mass energy (ρ) [5], thus the components become

$$T^{\mu\nu} \approx \left(\begin{array}{c|c} \rho & \rho v^j \\ \hline \rho v^k & \rho v^j v^k + p\delta^{jk} \end{array}\right) \tag{9}$$

Note that $u^0 = \gamma \approx 1$ in the Newtonian case. Then the conservation of energy and momentum will appear if $T^{\mu\nu}$ is contracted with ∂_ν

$$\partial_\nu T^{\mu\nu} = 0. \tag{10}$$

This immediately gives for $\mu = 0$

$$\partial_\nu T^{0\nu} = \frac{\partial p}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{11}$$

the *equation of continuity*. The $\mu = k$ components become:

$$\frac{\partial}{\partial t}(\rho v^k) + \frac{\partial}{\partial x^j}(\rho v^j v^k) + \frac{\partial}{\partial x_k}p = 0 \tag{12}$$

Rewriting this equation and using the equation of continuity, we get the following set of equations:

$$\begin{aligned} \rho\frac{\partial}{\partial t}v^k + v^k\frac{\partial}{\partial t}\rho + v^k\frac{\partial}{\partial x^j}(\rho v^j) + \rho v^j\frac{\partial}{\partial x_j}v^k &= -\frac{\partial}{\partial x^k}p \\ -v^k\frac{\partial}{\partial t}\rho - v^k\frac{\partial}{\partial x^j}(\rho x^j) &= 0 \end{aligned}$$

This finally gives *Euler's equation* :

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p \tag{13}$$

In general relativity the ∂_ν is replaced by ∇_ν, so we get the following list of properties

[4] the following discussion is in a viewpoint of special relativity. However the equations can be generalised (to general relativity)

[5] because I took the speed of light 1, for a fair compairison the units must be equal and ρ should be multiplied with c^2

Properties of the energy momentum tensor

- $T^{\mu\nu}$ is the energy-momentum (or stress-energy) tensor, in which the momentum and energy are contained.
- $T^{\mu\nu}$ is symmetric: $T^{\mu\nu} = T^{\nu\mu}$
- $\nabla_\mu T^{\mu\nu} = 0$. An expression for the conservation of energy and momentum.

4 The Riemann Curvature Tensor

The defenition of the Riemann curvature tensor (in a coordinate basis) is

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma T^\alpha_{\beta\delta} \partial_\delta T^\alpha_{\beta\gamma} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \tag{14}$$

where the Γ's are the *connection coefficients*, sometimes refered to as *affine connections* . The need to introduce these terms comes from the generalization of the partial derivative, ∂_μ, to the *covariant derivative* ∇_μ. This is the proper derivative in general relativity, because it transforms as a tensor [6] . In flat spacetime, the covariant derivative reduces to the partial. Its definition is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda \tag{16}$$

Whereas the connection coefficients itself, can be defined with the additional condition called *metric compatibility*

$$\nabla_\rho g_{\mu\nu} = 0 \tag{17}$$

This condition also allows for lowering and raising indices of the metric. The connection coefficient are then defined by

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{18}$$

The connection coefficients and thus the Riemann Tensor are derived from the metric. Furthermore with metric compatibility contracting a covariant derivative simplifies

$$g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho V_\mu \tag{19}$$

Thus the metric can be interchanged with the (covarient) derivative.

Now the relevance of the Riemann Tensor is its property that it is the only tensor that includes the metric and is linear in its second derivatives. $\partial_\mu \partial_\nu g^{\alpha\beta}$, which would be a logical choice, is not right, because the result doesn't represent a proper tensor (i.e. it doesn't transform as a tensor). $\nabla_\mu \nabla_\nu g^{\alpha\beta}$ is wrong as well, because this is zero by metric compatibility. Because of (18) the Riemann tensor in its covariant form ($R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^\mu{}_{\beta\gamma\delta}$) reduces to

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(\partial_\gamma \partial_\beta g_{\alpha\delta} - \partial_\gamma \partial_\alpha g_{\delta\beta} - \partial_\delta \partial_\beta g_{\alpha\gamma} + \partial_\delta \partial_\alpha g_{\gamma\beta}\right) \tag{20}$$

As has been said before, the Riemann Tensor is composed of 256 components, but all are not independent. The following symmetries from the equation above reduce the independent

[6] That ∂_μ doens't transform as a tensor is easily seen:

$$\partial_{\mu'} V^{\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu\right) \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu V^\nu\right) + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\mu} V^\mu \tag{15}$$

The first part obeys tensor transformation, but the second term obviously does not.

components of the Riemann Tensor to 20. [7]

symmetries of the Riemann tensor

$$\begin{aligned}
&\text{(antisymmetry on last two indices)} & R_{\alpha\beta\gamma\delta} &= R_{\alpha\beta[\gamma\delta]} && \text{(21a)}\\
& & R_{\alpha[\beta\gamma\delta]} &= 0 && \text{(21b)}\\
&\text{(antisymmetry on first two indices)} & R_{\alpha\beta\gamma\delta} &= R_{[\alpha\beta]\gamma\delta} && \text{(21c)}\\
&\text{(symmetry under pair exchange)} & R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} && \text{(21d)}\\
&\text{(completely vanishing of antisymmetric part)} & R_{[\alpha\beta\gamma\delta]} &= 0 && \text{(21e)}
\end{aligned}$$

The '[ ]' is a notation for:

$$R_{\alpha\beta[\gamma\delta]} = \frac{1}{2}\left(R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\delta\gamma}\right) \tag{22}$$

$$R_{\alpha[\beta\gamma\delta]} = \frac{1}{3!}\left(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} - R_{\alpha\beta\delta\gamma} - R_{\alpha\gamma\delta\beta} - R_{\alpha\delta\gamma\beta}\right) \tag{23}$$

Thus a '+' sign for an even permutation of $\beta\gamma\delta$, and a '-' sign for an odd permutation of $\beta\gamma\delta$. From (22) it is easily seen where the antisymmetry comes from

The Riemann tensor is of fourth order, but can be contracted to form the *Ricci tensor* $R^{\mu\nu}$ or the *curvature scaler* R.

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{24}$$

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{25}$$

These are needed to form the Einstein tensor $G^{\mu\nu}$.

5 The Einstein tensor

The Einstein Tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{26}$$

properties of the einstein tensor

- $G_{\mu\nu}$ vanishes when spacetime is flat
- $G_{\mu\nu}$ is constructed from the Riemann tensor and the metric.
- $G_{\mu\nu}$ is different from other tensors that can be constructed from the Riemann tensor and the metric by the demands:

  (1) $G_{\mu\nu}$ is linear in Riemann

  (2) $G_{\mu\nu}$ (like $T_{\mu\nu}$) is symmetric

[7] Not all these relations are independent from each other, however. (21a) and (21b) follow directly from the Riemann Tensor. Relation (21c) follows when the metric is included. (21d) and (21e) follow from the first three equations

(3) $G_{\mu\nu}$ (like $T_{\mu\nu}$) obeys the *Bianchi identity* [8] [9]

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{30}$$

6 Einstein's modification

Einstein modified his Field equation in order to provide a static universe. He introduced the term Λ as cosmological constant

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu} \tag{31}$$

A nonzero value of Λ no longer results in the vanishing of $G^{\mu\nu}$ in vacuum. I will not elaborate on the cosmological constant, but give an (simple) application of the Einstein equations in cosmology.

7 Applications on cosmology

The Robertson-Walker metric[10] is (without derivation)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{32}$$

and for the inverse $g^{\mu\nu}$ (because $g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}_{\nu}$ and the metric is diagonal).

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix} \tag{33}$$

Where $a = a(t)$ is a time-dependent scale factor. k measures the curvature and is only interesting for $k = -1,\ k = 0,\ k = +1$. The table gives an qualitive relation between the

[8] That the Bianchi identity is indeed true can best be seen in an inertial frame. Because $G_{\mu\nu}$ (like $T_{\mu\nu}$ is a proper tensor, the relation will then be true in every frame. In an inertial frame the connection coefficients vanish and the derivatives are partial. The Riemann tensor is then simplified to

$$R_{\alpha\beta\gamma\delta} = \partial_\beta \partial_\delta g_{\alpha\gamma} - \partial_\beta \partial_\gamma g_{\alpha\delta} + \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\alpha \partial_\delta g_{\beta\gamma} \tag{27}$$

And now contracting this ($g^{\alpha\lambda} R_{\alpha\mu\lambda\nu}$) and using property (19) we get for the Ricci Tensor and for the curvature scaler

$$R_{\mu\nu} = 0 - \partial_\mu \partial^\alpha g_{\alpha\nu} + \Box g_{\mu\nu} - \partial^\alpha \partial_\nu g_{\mu\alpha} \tag{28a}$$

$$R = g^{\mu\nu} R_{\mu\nu} = -\partial^\nu \partial^\alpha g_{\alpha\nu} + 0 - \partial^\alpha \partial^\mu g_{\mu\alpha} = -2\partial^\alpha \partial^\beta g_{\alpha\beta} \tag{28b}$$

Contraction of the Einstein tensor then gives

$$\partial^\mu G_{\mu\nu} = \partial^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = -\Box \partial^\alpha g_{\alpha\nu} + \Box \partial^\mu g_{\mu\nu} - \partial^\mu \partial_\nu \partial^\alpha g_{\mu\alpha} + \partial^\alpha (g_{\mu\nu} \partial^\mu) \partial^\beta g_{\alpha\beta}$$

The first two terms cancel, as do the last two terms.

[9] a more generalised version of the Bianchi Identity is

$$0 = \nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \tag{29}$$

where I used the antisymmetry of the Riemann tensor in its first two components (i.e. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$).

[10] The RW-metric demands two assumptions about the universe: Isotropy and homogeneity of space

curvature of the universe (k) and the density (Ω). The aim of this section is to derive this relation.

k	curvature		Ω
-1	negative	open	$\Omega < 1$
0	euclidean	flat	$\Omega = 1$
1	positive	closed	$\Omega > 1$

Because $g_{\mu\nu}$ is given, the connection coefficients can be calculated with (18). Some are explained in more detail.

$$\Gamma^0_{11} = \frac{1}{2}g^{0\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right) = -\frac{1}{2}g^{00}\partial_0 g_{11} = \frac{a\dot{a}}{1-kr^2} \tag{34a}$$

$$\Gamma^0_{22} = -\frac{1}{2}g^{00}\partial_0 g_{22} = a\dot{a}r^2 \qquad \Gamma^1_{11} = \frac{kr}{1-kr^2} \tag{34b}$$

$$\Gamma^0_{33} = -\frac{1}{2}g^{00}\partial_0 g_{33} = a\dot{a}r^2\sin^2\theta \tag{34c}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \tag{34d}$$

$$\Gamma^1_{22} = -\frac{1}{2}g^{1\rho}\partial_1 g_{22} = -\frac{1}{2}\frac{1-kr^2}{a^2}2r^2a^2 = -r(r-kr^2) \tag{34e}$$

$$\Gamma^1_{33} = -r(r-kr^2)\sin^2\theta \qquad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \tag{34f}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{34g}$$

where $\dot{a}$ is the time derivative of a(t). The nonzero components of the Ricci-Tensor are:

$$\begin{aligned} R_{00} &= R^\lambda{}_{0\lambda 0} = \partial_\lambda\Gamma^\lambda_{00} - \partial_0\Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\mu\lambda}\Gamma^\mu_{00} - \Gamma^\lambda_{\mu 0}\Gamma^\mu_{0\lambda} = \\ &= -\frac{\partial}{\partial t}\left(\frac{3\dot{a}}{a}\right) - 3\frac{\dot{a}^2}{a^2} = 3\frac{\ddot{a}}{a} \end{aligned} \tag{35a}$$

$$\begin{aligned} R_{11} &= R^\lambda{}_{1\lambda 1} = \partial_0\Gamma^0_{11} - \partial_1\Gamma^1_{11} - \partial_1\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) \\ &+ \Gamma^0_{11}\left(\Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}\right) + \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) \\ &- \Gamma^1_{01}\Gamma^0_{11} - \Gamma^0_{11}\Gamma^1_{10} - \Gamma^1_{11}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^2_{11} - \Gamma^1_{13}\Gamma^3_{13} = ... \\ &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \end{aligned} \tag{35b}$$

$$R_{22} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) \tag{35c}$$

$$R_{33} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\sin^2\theta \tag{35d}$$

With the curvature scalar becoming

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right) \tag{36}$$

Now we apply the Einstein equation (1) with $T_{\mu\nu}$ for a perfect fluid (8) in a comoving frame. Then

$$8\pi G T^\mu{}_\nu = 8\pi G\ \mathrm{diag}(\rho, -p, -p, -p) = G^\mu{}_\nu \tag{37}$$

The two independent components of the Einstein Tensor are

$$G^0{}_0 = R^0{}_0 - \frac{1}{2}Rg^0{}_0 = g^{00}R_{00} - \frac{1}{2}R\delta^0_0 = \frac{3\dot{a}^2 + 3k}{a^2} \tag{38a}$$

$$G^1{}_1 = R^1{}_1 - \frac{1}{2}Rg^1{}_1 = g^{11}R_{11} - \frac{1}{2}R\delta^1_1 = \frac{-2a\ddot{a} - \dot{a}^2 - k}{a^2} \tag{38b}$$

Equation (38a)[11] leads to *Friedmann equation*

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = -k \tag{39}$$

With the following notations

$$H = \frac{\dot{a}}{a} \qquad \rho_c = \frac{3H^2}{8\pi G} \qquad \Omega = \frac{\rho}{\rho_c}$$

This can be rewritten as

$$\boxed{\Omega - 1 = \frac{k}{a^2 H_0^2}} \tag{40}$$

The relation between Ω and k is clear.

Of course many more interesting relations can be obtained, but this would be too much for this paper. For those interested I would suggest the college cosmology.

[11](38b) leads to the equation of conservation of energy. I will not elaborate on this equation in this paper, but it does play an important role in cosmology (e.g. to define the decelaration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2}$)

References

[1] Misner C.W., Thorne K.S., Wheeler J.A. (1973) *Gravitation*

[2] Weinberg, S. (1972) *Gravitation and Cosmology*

[3] Peacock, J.A. (1999) *Cosmological physics*

[4] Sean M. Carroll (1997) *Lecture Notes on General Relativity*: part 1 (no-nonsense introduction), part 3 (curvature), part 4 (gravitation) and part 8 (cosmology), http://pancake.uchicago.edu/c̃arroll/notes/

Index